CONTINUA OF MINIMUM CAPACITY[1]

G. C. EVANS

1. **Surfaces containing a given volume.** In an endeavour to simplify a proof of Liapounoff [2], to the effect that in the problem of the forms of equilibrium of rotating liquids the sphere would be the only form for a liquid at rest, Poincaré [1] was led to the consideration of electric capacities of solids of given volume, and arrived at the result that among such bodies the sphere would have minimum capacity. The present paper originated in the question of the determination of the surface sheet, without volume, which would be bounded by a given closed curve in space, and, among all such surfaces, have minimum capacity.

In the discussion of his problem, Poincaré assumes tacitly that there do exist one or more bodies of the given volume, with smooth boundaries, which furnish relative minima for the capacity with respect to neighboring forms; and his treatment amounts to a proof that among these the sphere furnishes the absolute minimum.

Let the body F, with smooth exterior boundary S, be considered as a conductor on which a positive charge m is spread so as to be in equilibrium—that is, the charge lies entirely on the surface S with a surface density $\sigma(P)$, and its potential $V(M)$ has a constant value V_0 within S, is continuous across S, satisfies Laplace's equation $\nabla^2 V=0$ outside S and vanishes at infinity. The density on S is given by the equation

$$\sigma(P) = -\frac{1}{4\pi}\frac{dV}{dn}, \qquad n \text{ the exterior normal.}$$

The energy of the distribution may be written in the equivalent forms

$$(1)\quad I = \frac{1}{8\pi}\int_W (\nabla V)^2 dP = \frac{1}{8\pi}\int_W (V_x^2 + V_y^2 + V_z^2)dP,\ W = \text{entire space},$$

$$(2)\quad I = \frac{1}{2}\int_S \sigma(P)V_0 dP = \tfrac{1}{2}mV_0.$$

The capacity of K of F is defined by the equation

$$(3)\qquad KV_0 = m \quad \text{or} \quad I = \frac{m^2}{2K} = \tfrac{1}{2}KV_0^2.$$

[1] Presidential address presented to the Society, January 1, 1941, under the title *Surfaces of minimum capacity*.

According to Poincaré, the variation δI of I due to a displacement of the points of S, along normals, of amount $\delta n(P) = \epsilon\psi(P)$ is given by the equation

$$(4) \qquad \delta I = -\frac{1}{8\pi}\int_S \delta n(P)(\nabla V)^2 dP = -\frac{\epsilon}{8\pi}\int_S \psi(P)\left(\frac{dV}{dn}\right)^2 dP,$$

the integral of the variation of the function $V(P)$ itself being negligible since the conductor distribution is in equilibrium. That (4) thus follows from (1) is not altogether evident. It may be justified, however, in the following way.

Let S_1 be the displaced surface, P_1, Q_1 points on S_1 corresponding to points P, Q on S. Since for a given total mass m on S the energy for a conductor distribution is a minimum, I will have a stationary value for small changes of the distribution $m(\epsilon)$ of m, and we may neglect such changes. Hence we may assume the relation

$$\Delta I = \frac{1}{2}\int_S dm(\epsilon_Q)\int_S \left(\frac{1}{P_1Q_1} - \frac{1}{PQ}\right) dm(\epsilon_P),$$

the potential at an arbitrary point M being given by the formula

$$V(M) = \int_S \frac{\sigma(P)}{MP}\, dP = \int_S \frac{dm(\epsilon_P)}{MP}.$$

We substitute

$$\frac{1}{P_1Q_1} - \frac{1}{PQ} = \left(\frac{1}{P_1Q_1} - \frac{1}{P_1Q}\right) + \left(\frac{1}{P_1Q} - \frac{1}{PQ}\right)$$

and change the order of integration in the integral of this second parenthesis, at the same time interchanging P with Q and P_1 with Q_1 so that we shall have

$$\Delta I = \frac{1}{2}\int_S \sigma(Q)dQ\int_S \left\{\left(\frac{1}{P_1Q_1} - \frac{1}{P_1Q}\right) + \left(\frac{1}{PQ_1} - \frac{1}{PQ}\right)\right\}\sigma(P)dP.$$

We denote by $V_1(Q')$ the potential of the displaced mass at some point Q' between Q and Q_1, and by $V(Q'')$ the potential of the original mass at some point Q'' between Q and Q_1. Then

$$\Delta I = \frac{\epsilon}{2}\int_S \sigma(Q)\psi(Q)\left\{\left(\frac{dV_1}{dn}\right)_{Q'} + \left(\frac{dV}{dn}\right)_{Q''}\right\} dQ.$$

We assume that S and S_1 are sufficiently smooth so that $\sigma(Q)$ is bounded and so that the derivatives of the potential approach the

limits given by the usual well known formulae, as ϵ tends to zero. Then

$$(5)\qquad \begin{aligned} \lim_{\epsilon\to 0}\left(\frac{dV}{dn}\right)_{Q''} &= \mp 2\pi\sigma(Q) + \int_S \frac{\cos(n_Q, QP)}{PQ^2}\sigma(P)dP, \\ \lim_{\epsilon\to 0}\left(\frac{dV_1}{dn}\right)_{Q'} &= \pm 2\pi\sigma(Q) + \int_S \frac{\cos(n_Q, QP)}{PQ^2}\sigma(P)dP, \end{aligned}$$

the upper sign being employed when $\psi(Q)>0$, the lower when $\psi(Q)<0$. Hence, for $\delta I=\epsilon \lim_{\epsilon\to 0}(\Delta I/\epsilon)$, we have the value

$$(6)\qquad \delta I = \epsilon\int_S \sigma(Q)\psi(Q)dQ\int_S \frac{\cos(n_Q, QP)}{PQ^2}\sigma(P)dP.$$

Inside the conductor the potential is constant, so that as a point approaches S, from inside, the value dV/dn_- is zero. Accordingly, by combining the first two of equations (5), we have

$$\frac{dV}{dn_+} = -4\pi\sigma(Q) = 2\int_S \frac{\cos(n_Q, QP)}{PQ^2}\sigma(P)dP.$$

Thus the formula (6) for δI reduces to the value given by (4).

In order that the capacity be a minimum, with given volume, it is necessary that $\delta I=0$, where δI is given by (4) and $\psi(P)$ is subject to the condition

$$\int_S \psi(P)dP = 0.$$

It follows therefore that $(dV/dn_+)^2$ is constant on S, and since $dV/dn_+=-4\pi\sigma$, we have

$$(7)\qquad \sigma(P) = \text{const. on } S.$$

Moreover, since now σ is constant on S, we may write the formula for δI in the form (the total mass m of course remaining unchanged)

$$\delta I = -\frac{S^2\sigma^2}{2K^2}dK = -2\pi\sigma^2 d\tau$$

in which the volume is changed, and $d\tau$ and dK are corresponding differential changes of volume and capacity. For such changes, then,

$$\frac{S^2 dK}{K^2} = 4\pi d\tau.$$

Suppose now that the body is deformed so as to remain similar to itself. Its capacity will change in the ratio of corresponding lengths, so that

$$\frac{dK}{K} = \frac{1}{3}\frac{d\tau}{\tau},$$

and, in the case of bodies for which (7) is satisfied,

$$K = \frac{1}{12\pi}\frac{S^2}{\tau}.$$

The sphere satisfies the condition (7), and it has the smallest S for a given τ. Hence if there is any smooth S which has the minimum capacity for a given volume, the sphere is that body.

Incidentally, we notice that if we make no requirement about the volume, but consider the surface merely as a sheet or cap, bounded, for instance, by a given closed curve in space, the requirement that the capacity be a minimum implies, if S be sufficiently smooth, that δI be zero, as given by (6). Since there is no restriction on $\psi(Q)$ in the neighborhood of any point of S, not on the boundary, we have the relation

$$(8) \qquad \int_S \frac{\cos(n_Q, QP)}{PQ^2}\sigma(P)dP = 0 = \int_S \frac{\cos(n_Q, QP)}{PQ^2}dm(e_P).$$

By means of the first of (5) the equation (8) may be written in the form

$$(8') \qquad \frac{dV}{dn_+} + \frac{dV}{dn_-} = 0,$$

where, of course, we still have the relation

$$(8'') \qquad \frac{dV}{dn_+} - \frac{dV}{dn_-} = -4\pi\sigma.$$

2. **Capacity and logarithmic potential.** In the discussion of the stability of the form of a rotating liquid, and in particular the spherical form of a liquid at rest, Liapounoff treated the second variation of the energy integral and showed it to be positive. Poincaré endeavored, as we have mentioned, to provide a simpler treatment for this special case. But in order to complete the problem in a satisfactory manner without assuming the existence of a smooth form for which capacity is a minimum further details are necessary. The possibility of a fuller

use of fundamental inequalities for this purpose was noticed by Faber [3] and Szegö [5]. This use is sufficiently illustrated in some corresponding, but perhaps more difficult, problems in the plane, and accordingly we turn to those.

For the plane, as in space, the capacity is to be regarded as an index of the charge which is necessary in order to raise the body to a given constant potential V_0. The equation $V_0 = m/K$, however, if carried over to the logarithmic potential has the disadvantage of yielding infinite and negative capacities. A more satisfactory definition, especially from the point of view of the complex variable, is given by

$$V_0 = m \log \frac{1}{K}; \qquad K = e^{-V_0/m}. \tag{9}$$

This is the one which we adopt. It is equivalent to the "transfinite diameter." By means of it, capacity is of dimension one in length in the plane (as in space), with regard to transformations of similarity, and the capacity of a circle (like that of a sphere) is equal to its radius.

Given an arbitrary bounded closed set F in the plane it may not be possible to distribute the mass m on F in such a way that the potential will remain constant everywhere on F because the frontier s which bounds the infinite domain T exterior to F may not consist entirely of points which are regular, with respect to T, for the Dirichlet problem with continuously given boundary values on s—for example, if s has isolated points. We can however approximate to T by a nested sequence of domains the boundary of each of which consists of a finite number of closed simple analytic curves, and thus define K by the obvious limiting process. In particular, if for a given m the potential V_0 does not remain bounded in the limit, the capacity of F is zero.

Accordingly we may limit ourselves to the sets F whose exterior frontiers consist only of points regular with respect to the Dirichlet problem. Let $u(M)$ be the potential of unit mass on F distributed so as to have constant potential V_0 on F. Then, with $r = OM$ where O is any fixed point,

$$u(M) = \log \frac{1}{r} + H(M) = -\log r + H(M)$$

with $H(M)$ a function which is harmonic in T and vanishes at infinity. Consequently the function (a special case of the Green's function)

$$V(M) = V_0 - u(M),$$

becomes infinite like $\log r$ at ∞ and vanishes on F. In fact,

$$V(M) = \log r - G(M), \qquad G(M) = H(M) - V_0, \tag{10}$$

where $G(M)$ is harmonic in T, and regular at ∞, at which point it takes on the value $G(\infty) = -V_0$. We have therefore

$$K = e^{G(\infty)}. \tag{11}$$

We follow now the procedure as sketched briefly by Szegö for analogous problems [5]. Consider the level curves $V(M)=\lambda$ of the function $V(M)$. For $\lambda>0$ these curves have no points in common with F since the points of the boundary s are regular; except for a finite number of values of λ, provided that λ is bounded away from 0, the locus $V(M)=\lambda$ consists of a finite number of simple closed analytic curves [6, b], and for large values of λ of a single piece. We denote this locus (and also its total length) by C_λ, the total area inside it by A_λ and the exterior normal direction by n. Thus $\lim_{\lambda\to 0} A_\lambda = \text{meas. } F$.

We have

$$2\pi = \int_{C_\lambda} \frac{dV}{dn}\, ds, \qquad \frac{dA_\lambda}{d\lambda} = \int_{C_\lambda} \frac{dn}{dV}\, ds,$$

the first equality being an expression of the fact that V is harmonic except for the additive term $\log r$. But by Schwarz's inequality and the isoperimetric inequality applied successively, we have

$$\begin{aligned} 2\pi \frac{dA_\lambda}{d\lambda} &= \int_{C_\lambda} \left(\left(\frac{dV}{dn}\right)^{1/2}\right)^2 ds \cdot \int_{C_\lambda} \left(\left(\frac{dn}{dV}\right)^{1/2}\right)^2 ds \\ &\geqq \left(\int_{C_\lambda} ds\right)^2 = C_\lambda^2 \geqq 4\pi A_\lambda. \end{aligned} \tag{12}$$

Thus $(1/A)dA/d\lambda \geqq 2$, and

$$\log A_\lambda - \log A_{\lambda_1} \geqq 2(\lambda - \lambda_1).$$

Hence if we let $\lambda_1 \to 0$, we have

$$A_\lambda \geqq (\text{meas. } F)e^{2\lambda}.$$

In this expression we now let $\lambda \to \infty$. For large values of λ we may use polar coordinates, and write, since $r = e^\lambda e^{G(M)}$,

$$A_\lambda = \frac{1}{2}\int_0^{2\pi} r^2 d\theta = \tfrac{1}{2}e^{2\lambda}\int_0^{2\pi} e^{2G(M)} d\theta.$$

Consequently

$$\frac{1}{2}\int_0^{2\pi} e^{2G(M)}d\theta \geqq \text{meas. } F;$$

and if we let $\lambda \to \infty$ so that $G(M) \to G(\infty)$, we shall have

$$\lim_{\lambda\to\infty} \int_0^{2\pi} e^{2G(M)}d\theta = 2\pi K^2.$$

Hence

$$\pi K^2 \geqq \text{meas. } F. \tag{13}$$

The equality sign in (12) can hold only if it holds identically in λ; that is, by Schwarz's inequality, only if dV/dn and dn/dV are proportional on each C_λ so that dn/dV is a function of λ and the C_λ form a set of parallel curves, and by the isoperimetric inequality, only if the C_λ are circles. In this case F is a circle and its radius is K. Since the difference of the two sides of (13) represents the integral from 0 to ∞ of a not negative quantity, the same result holds for the equality sign in (13).

The inequality (13) holds for any bounded closed set F. In fact, both sides of the inequality are obtained by the same process in terms of a nested sequence of approximate domains. But other sets besides the circle, if F is not a single continuum, satisfy the relation $\pi K^2 = F$.

3. **Applications of conformal mapping.** Except for a rotation and translation, a conformal transformation which maps the exterior of a bounded continuum F in the z plane into the exterior of a circle of radius r in the w plane, without distortion at ∞, may be written in the form

$$z - w = \sum_1^\infty \alpha_n w^{-n}. \tag{14}$$

The radius r is called the Robin constant. We find that $r = K$, where K is the capacity of F. This is seen immediately by comparing (14) with the transformation

$$w = re^{V(z)+iU(z)} \tag{15}$$

where $U(z) = U(M)$ is the function conjugate to $V(z) = V(M)$, and by noting the value of $|dw/dz|$ in (15) at ∞.

A direct calculation from (14) in this case shows that

$$\text{meas. } F = \pi r^2 - \pi \sum_1^\infty n r^{-2n} |\alpha_n|^2,$$

so that in particular meas. $F \leqq \pi r^2$, which is a special case of (13), and

$$|\alpha_1| \leqq r^2. \tag{16}$$

On the other hand, if A is an interior domain in the Z plane, containing the origin, which is mapped conformally on the interior of a circle $|W| = R$ so that $Z = 0$ goes into $W = 0$ and there is no distortion at $Z = 0$, the transformation, except for a possible rotation, may be written in the form

$$Z = W + \sum_{2}^{\infty} \beta_n W^n. \tag{17}$$

Hence

$$A \geqq \pi R^2 + \pi \sum_{2}^{\infty} n |\beta_n|^2 R^{2n} \geqq \pi R^2.$$

By means of the transformations

$$w = \frac{1}{W}, \qquad z = \frac{1}{Z}$$

the situation is reduced to a transformation of exterior domains, in which $r = 1/R$. But the transformation of these exterior domains is given by

$$z = \left(\frac{1}{w} + \sum_{2}^{\infty} \beta_n w^{-n}\right)^{-1} = w(1 + \beta_2 w^{-1} + \beta_3 w^{-2} + \cdots)^{-1}$$
$$= w(1 - \beta_2 w^{-1} + \cdots).$$

Consideration of the resulting univalent relation between branches of $z' = z^{1/2}$ and $w' = w^{1/2}$ yields immediately, by (16), the relation

$$|\beta_2| \leqq \frac{2}{R}.$$

The above results, which are involved in a direct comparison of the coefficients of the power series are of course well known. But also other known results are obtained in terms of them by means of elementary statements about capacity. From the definition of capacity and the fact that V_0 is the potential due to a distribution of positive mass, there follows easily the statement that if F_1 is contained in F_2, or is the orthogonal projection of F_2 on a line, then $K(F_1) \leqq K(F_2)$. By means of the conformal transformation[2]

[2] This is the base transformation for the study of aerofoils.

$$w = \tfrac{1}{2}(z + (z^2 - a^2)^{1/2}) \tag{18}$$

in which $w=(1/2)(x+(x^2-a^2)^{1/2})$ when $z=x$ is real and $>a$, the region exterior to the segment $(-a, 0)$, $(a, 0)$ is transformed into the exterior of the circle $|w| = a/2$, without distortion when $z=\infty$. Hence the capacity of a line segment of length $2a$ is $a/2$. Moreover the capacity of a segment of a straight line of length l is less than that of a collection of segments of the line of the same total length l.

Let p be the projection, of linear measure l, on an arbitrary straight line, of the closed bounded set F. Then $K(p) \geqq l/4$ and $K(p) \leqq K(F)$. Hence

$$l \leqq 4K(F). \tag{19}$$

The theorem expressed by (19) was proved in slightly less general form by Pólya, by the application of Tchebytchef polynomials in the definition of transfinite diameter [4].

Consider again an interior domain and the transformation (17). Let d be the minimum distance from O to a point of the boundary of A, R the radius of the corresponding circle in the W plane, there being again no distortion at the origins, which correspond. We have immediately the Koebe result: $d \geqq R/4$. In fact, there is a point P on the boundary of A such that $OP=d$. Its image P' is a point of F, where F is the complement of the infinite domain obtained from A by the transformation $z=Z^{-1}$, and the capacity $K(F)$ is the radius $r=1/R$ of the corresponding circle in the w plane, with $w=W^{-1}$. But, denoting OP' by d', we have from (19)

$$\frac{1}{d} = d' \leqq 4K(F) = 4r = \frac{4}{R},$$

from which the Koebe result follows. That $R/4$ is the best result obtainable follows from the consideration of a region in the z plane which is a narrow approximation to a straight segment.

Again, of all the circles C contained in A and containing O, the radii have an upper bound l, and there is a circle of radius l, whose interior lies in A and such that O lies in the circle or on its circumference. If this situation is interpreted in terms of the transformations $z=Z^{-1}$, $w=W^{-1}$ we find a circle C', of radius l', circumscribed about F, but not necessarily the smallest one.

The smallest circle C'' which circumscribes F has a radius l'' which is $\leqq \eta/3^{1/2}$, where η is the diameter of F. But $\eta \leqq 4K(F)$, so that $l'' \leqq 4r/3^{1/2}$. The image of C'' in the z plane is a circle C in A and containing O in its interior. Let P, Q be opposite ends of a diameter of C

through O. Then P', Q' are opposite ends of a diameter of C'' through O, and

$$l \geqq \frac{1}{2}(PO + OQ) = \frac{1}{2}\left(\frac{1}{P'O} + \frac{1}{OQ'}\right) \geqq \frac{1}{l''},$$

since the minimum value of $(1/P'O)+(1/OQ')$ is obtained when $P'O = OQ'$. Accordingly

$$l \geqq \frac{3^{1/2}}{4r} = \frac{3^{1/2}}{4} R.$$

The value $.43R$ is not, however, the best value which can be obtained for the lower bound of l. By means of stricter inequalities on the coefficients of the Taylor series R. M. Robinson obtains the value $\frac{1}{2}R$. And if the requirement that O shall be contained in the circle is dropped, the radius of the largest circle which lies in A is at least as great as a number k, which as Landau shows [7] satisfies the inequality $k \geqq .55R$, and as Robinson shows [8] satisfies also the inequality $k \leqq .66R$.

4. **Open curves of minimum capacity.** The statement (19) about the capacity of the projection of F may be reworded as follows:

Of all continua which contain two fixed points A, B, the straight segment AB has the minimum capacity.

This theorem involves a new orientation towards the question, for it suggests the discussion of the continuum of minimum capacity which contains a given arbitrary collection of fixed points, finite in number. Pólya has considered this problem [4] and obtained the solution of it in the case of $2n$ points spaced regularly, n on each of two concentric circumferences, the desired continuum consisting in this case of the radial lines drawn from the points to the center of the circle.

The situation for n arbitrary points is discussed by Grötsch [9, b]. The aspects of the problem which are involved are seen in the case of three points in the z plane. Consider a circumference of radius r in the w plane, and on it three points $w_1 = r+0i$, $w_2 = re^{(\phi_1+\phi_2)i}$, $w_3 = re^{-(\phi_1+\phi_2)i}$ with ϕ_1, ϕ_2, ϕ_3 positive angles of sum π. A conformal transformation, with unit magnification at ∞, of the exterior of the unit circle into the exterior of some continuum C in the z plane exists, in which pairs of symmetrical points on the arcs $[0, \phi_1]$, $[0, -\phi_1]$, respectively, adjacent to w_1, go into single points of the Z plane; and similarly for the arcs adjacent to w_2 and w_3. The three points $re^{i\phi_1}$, $re^{-i\phi_1}$, $re^{i(\phi_1+2\phi_2)}$ thus go into one point z', and the points w_1, w_2, w_3

correspond to certain distinct non-collinear points z_1, z_2, z_3. In fact, collinear points z_1, z_2, z_3 would arise only if one of the arcs ϕ_1, ϕ_2, ϕ_3 were zero, as is suggested by (18). The continuum C thus consists of three analytic rays from z' to z_1, z_2, z_3.

It is proved that C has the least capacity of any continuum which contains the three points z_1, z_2, z_3 regarded as fixed, by reducing the question to the consideration of the maximum sum of "modular ratios" for an area [9, a]. This number is in fact equal to the upper bound of the Dirichlet integral of a function u summed over strips into which an area is subdivided, where at the ends of the strips u has the values 0 and 1 respectively, and along the sides of the strips $du/dn=0$, and u satisfies Laplace's equation in each strip. It is then shown by examination of the parameters involved in the transformation, with reference to a principle of continuity in the variation of those parameters, that the above treatment is equivalent to that of the general case. The situation with n points is treated in the same manner, except that intermediate arcs, not abutting on any w_i, arise from the restrictive nature of the conformal transformation.[3]

5. **Open surfaces of minimum capacity.** In three dimensions, the simplest problem analogous to those just treated would be to prove the existence of a surface sheet, bounded by a given closed curve in space, which among all such surfaces would have minimum capacity [10, a, b]. This problem would be the counterpart of the simplest plane problem, namely, the continuum of minimum capacity containing two given points. But it may be of interest for the record that the author put to himself and solved the space problem first, before becoming familiar with the problem in the plane, and the latter has been introduced in this address, with its fundamental relation to conformal representation, mostly as an extension of the space problem rather than the reverse. In fact, it will be seen later that the statements of the two problems do not exactly correspond.

We consider a simple closed space curve s, itself of zero capacity, in our three-dimensional euclidean space B and topologically equivalent to a circle in the sense that there exists a one-one continuous transformation ξ of a sufficiently large sphere containing s into itself, in such a way that s goes into a circle σ interior to the sphere. For convenience, we suppose that the points of the surface of the sphere are unchanged by the transformation, and we may regard ξ as extended outside the sphere as the identical transformation.

[3] The author apologizes for the skimpy treatment of Grötsch's important contribution. Lack of space requires him to refer the reader to the cited memoirs.

Let now S be a piece of surface consisting of a closed set of points which contains s and which, except in the neighborhood of s, consists of a finite number of regular surface elements, each surface element having bounded curvatures. Then, without specifying further for the moment the notion of s being the boundary of S, the analysis of §1 applies, and at every point in a surface element, not on an edge of the element or on s, the conductor potential of S will satisfy the relation (8′). In order, however, not to have to distinguish between local positive and negative normal directions at points of S we shall consider normal directions n_1, n_2 as positively directed away from S and write (8′) and (8″) in the form

$$\frac{dV}{dn_1} = \frac{dV}{dn_2}, \tag{20}$$

$$\frac{dV}{dn_1} + \frac{dV}{dn_2} = -4\pi\sigma. \tag{20′}$$

The function $V(M)$ has a constant value, say 1, on each surface element, and its derivatives are continuous as we approach an interior point of such an element [9, a, p. 165]. If then we define $v_1(M)$ locally as equal to V on the side 1 of the surface element and as equal to $2-V$ on the side 2 of that element, the function $v_1(M)$ and its first partial derivatives will be continuous across the element, on account of (20); and $v_1(M)$ will be harmonic at points of the element. Consequently the surface element itself will be a piece of analytic surface, being a level surface, without singularities, of a harmonic function. Similarly an extension of V from the side 2 to the side 1 yields a harmonic function v_2.

It may be shown that there exists a two-valued harmonic function $v(M)$ which has the following properties:

(a) Either branch of the function is harmonically extensible along any curve in space which does not meet s.

(b) The function is bounded and one of its two values tends to 0 at ∞, the other to 2.

(c) If the two values $v_1(M)$, $v_2(M)$ of v are distinct, a closed path which starts from M and loops s once, returning to M, carries $v_1(M)$ into $v_2(M)$, and vice versa, whereas if the closed path does not loop s it carries each value back into itself.

(l) At every point M not on s, $v_1(M)+v_2(M)=2$.

(m) Every point of s is a limit point of the locus $v(M)=1$.

We obtain the function $v(M)$ by applying the Schwarz alternating process (in a form which is equivalent to the sweeping out process,

without its distributions of mass) to a three-dimensional Riemann multiple space $\mathfrak{M}$. In our case the curve s serves as a branch curve, analogous to a branch point in the complex plane, and the Riemann manifold may be constructed simply, by introducing the proper cut space. The necessary cut surface Σ may be drawn, for example, as the image in terms of the transformation ξ of a right circular cylindrical surface with base σ (in the xy plane) and extending to ∞ (in the direction of the positive z axis). The function $v(M)$ is to be single valued in each of the cut spaces. The double cut space may again be replaced by a finite number of simple overlapping spaces by replacing the cylindrical surface by truncated conical surfaces within and outside it, and by means of these overlapping regions the connections of the Riemann manifold may be made across the cut surface.

At ∞ in the first sheet of $\mathfrak{M}$ within the half cylinder Σ, v_1 is to be assigned the value 0 and outside it the value 2, whereas in the second sheet v_2 is to be assigned the value 2 at ∞ within the cylinder and the value 0 at ∞ outside it; and the Schwarz process is employed to draw the two functions v_1, v_2 together through the simple regions which overlap the cuts. There are however one or two difficulties, to expound the details of which would extend unduly the length of this paper. In the first place, the curve s is a curve of zero capacity and hence will not act as a carrier for boundary values, in spite of the fact that $v(M)$ is to assume the value 1 on s; and in the second place, the surface Σ and the pseudo-conical surfaces are merely the topological images of smooth surfaces and are therefore not necessarily free of sets of points which are irregular for the Dirichlet problem.

The first of these difficulties is overcome by replacing σ by a sequence of tori σ_n condensing to σ as $n \to \infty$. For each of the corresponding surfaces Σ_n in B a Riemann manifold $\mathfrak{M}_n$ is left, when the interior of Σ_n is cut out, and a function $v^{(n)}(M)$ is set up which has the given properties at ∞ and corresponds to the value 1 on the boundary Σ_n. The second difficulty is handled by making use of the fact that the generalized Dirichlet problem is solvable for domains in which points are simply counted and by performing the analysis in terms of functions which are superharmonic on the interior of the whole Riemann manifold. It is proved that the successive alternate functions are superharmonic even at the exceptional points of the cross cuts which define the overlapping regions.[4]

[4] Incidentally it may be noted that the above process for solving the generalized Dirichlet problem for a domain on a multiple space of a finite number of leaves enables one to arrive at the Green's function for the domain, which has at a given point a singularity of the form $1/r$ on only one of the leaves.

The sequence $\{v^{(n)}\}$ has a limit function $v(M)$ which is two-valued and of course harmonic in any simply counted bounded domain, so that it evidently satisfies (a). It is easily shown to satisfy (b) on account of the uniformity of convergence except in the neighborhood of s, and (c) and (l) similarly, since each function $v^{(n)}(M)$ has those properties. Because of the non-uniform convergence in the neighborhood of s the property (m) is established with a little more difficulty; it is proved by showing that in any neighborhood of a point of s there lies for each n at least one point P_n, distant from s by more than a given positive δ, where $v^{(n)}(P_n)=1$. The function $v(M)$ is not necessarily continuous at s, for s has not been restricted as to smoothness, but all of its points are limit points of the surface S which is the locus $v(M)=1$. It is clear that S occludes no points from ∞, that is, encloses no portion of space; in fact if the contrary were true we should have $v(M)\equiv 1$ in that portion, and therefore throughout space.

The facts just given about $v(M)$ lead to a more geometrical interpretation of the relation of s to S. Any simple closed curve which loops s once (having no points in common with s) has points in common with S. If in the neighborhood of S the curve is analytic, and cuts S at nonsingular points, it has an odd number of points in common with S, counting multiple intersections in the usual way.

We pass now to the converse statement. The two-valued harmonic function which satisfies the conditions (a), (b), (c) is unique.

The proof of this theorem lies essentially in an extension of Kellogg's uniqueness theorem, which may be stated as follows: Let $u(M)$ be harmonic and bounded on a domain Ω on a manifold $\mathfrak{M}$, which consists of a finite number of sheets, tending to 0 at ∞ if Ω is not a bounded domain, and let m be the upper bound of $u(M)$, $m>0$. Let F be the set of points Q on the boundary of Ω where $\limsup u(M)\geqq m-\epsilon$ as M tends to Q on one or another leaf of $\mathfrak{M}$. Then F considered as a set of points in ordinary space is of positive capacity. The details of the proof are not difficult, but must be omitted here on account of lack of space.

As a consequence of this theorem, since s is of zero capacity, the difference of two functions which satisfy (a), (b), (c) will have to have its upper and lower bounds on $\mathfrak{M}$ both equal to zero.

The surface S, thus determined as the level surface $v=1$ of the function v, furnishes the minimum capacity among all surfaces S' which have the following properties: S' is bounded and every point of s is a limit point of S'; except in the neighborhood of s, S' consists of a finite number of regular elements sufficiently smooth for an application of Green's theorem; every closed curve which loops s has a

point in common with S'. Thus S' is a cut surface for $\mathfrak{M}$. The proof depends upon a comparison of the energies of the conductor potential for S and of the conductor potential for S' for the same total mass, in terms of Green's theorem—a comparison which is made possible by evaluating the Dirichlet integral for the conductor potential of S first on $\mathfrak{M}$ and then on S'. The proof has already been given in detail [9, b].

6. **Open surfaces of a finite number of sheets.** The nature of our problem will be more clearly brought out by considering a slightly more general case, for example, the image s_1, s_2 in B of two non-linked circles σ_1, σ_2. In this case the connections of the Riemann manifold $\mathfrak{M}$ are easily imagined, since the branches of the two-valued function v are to be interchanged by analytic extension along any circuit which links s_1 or s_2 once, and thus unchanged by a circuit which links them both. If we denote by $u^{(1)}$, $u^{(2)}$ the two functions $v^{(1)}-1$, $v^{(2)}-1$ where $v^{(1)}$ and $v^{(2)}$ are the two-valued harmonic functions belonging to s_1, s_2 separately, the product $u^{(1)}u^{(2)}$ will be single valued on $\mathfrak{M}$; and also the function $u(M)$, defined by the equation

$$v(M) - 1 = u^{(1)}(M)u^{(2)}(M)u(M),$$

will be single valued in ordinary space. By expressing the fact that the Laplacian of the left-hand member is zero, and utilizing the fact that $1-u(M)$ vanishes canonically as M tends to ∞, and is represented by a potential, an integral equation may be found for the unknown function $u(M)$.

But it may also be remarked that if the curves s_1, s_2 are joined by a strip, arbitrarily narrow, so that s is made a simple space curve in the manner of §5, the capacity in the new situation becomes a limiting case of that of the old, since the capacity of the strip may be made arbitrarily small. In other words, the problem of finding a single continuum, or even a single surface, in three dimensions which contains both s_1 and s_2 is illusory; for the capacity of a joining curve may be made zero. Hence our problem refers essentially to the determination of surfaces, finite in number, which will be cut by any circuit which loops a component part of s. In two dimensions there is, however, also a second problem analogous to our three-dimensional one.

We close by discussing qualitatively some simple cases. If the curve s lies in a plane, the surface S is the plane surface enclosed. Nevertheless the surface S in general is evidently not a minimal surface. In fact, if s consists of two circles s_1, s_2 which are parallel cross sections of a long circular cylinder, at a considerable distance from each other,

the minimal surface would consist of the two parallel circular areas. On the other hand, it is evident that the equation (8) is not thus satisfied, so that the two flat discs do not constitute the arrangement for minimum capacity.

In the case of the Plateau problem, if the two circles just described are brought closer together, the discs remain a solution until there is actual contact; on the other hand, if the circles are withdrawn while they are bridged by the constricted longitudinal surface (the catenary of revolution) which satisfies the condition of zero mean curvature, this surface maintains itself until a central node is reached by means of the constriction. With the present problem there is only one surface at any stage, the unstable situation being avoided, in the continuous deformation, by the bulges in the disc-shaped surfaces which are required to minimize the energy. These are caused by the action at a distance which is expressed in terms of (8).

It is seen also by means of (8) that the surface S lies within the closed convex cover of s.

If the two circles of the previous paragraphs are connected along elements of the cylinder by means of a narrow strip, so that s becomes a simple curve, and the strip is twisted so that one of the circles is overturned, we have another figure which changes its character markedly as the circles are brought together. In fact, the surface changes continuously from a two-sided one when the circles are far apart, into a nodal surface and then finally into what is evidently a one-sided one when the circles are sufficiently close so that the curve s can be the boundary of a narrow Möbius strip.

The author must admit, in conclusion, that the methods described in this three-dimensional discussion are quite specialized, and it is hard to see how they can be used in a more general class of problems in the calculus of variations. Perhaps the introduction of a vector potential would offer some advantage for the general point of view.

References

1. H. Poincaré, *Sur un théorème de M. Liapounoff, relatif à l'équilibre d'une masse fluide*, Comptes Rendus de l'Académie des Sciences, vol. 104 (1887), pp. 622–625; also in *Figures d'Équilibre d'une Masse Fluide*, Paris, 1902.

2. A. Liapounoff, *Sur la stabilité des figures ellipsoidales d'équilibre*, Thèse, 1884, translation, Annales de la Faculté des Sciences de Toulouse, vol. 6 (1904), pp. 5–116.

3. G. Faber, *Beweis, dass unter allen homogenen Membranen von gleicher Fläche und gleicher Spannung die kreisförmige den tiefsten Grundton hat*, Bayerische Akademie der Wissenschaften, 1923, pp. 169–172.

4. G. Pólya, *Betrag zur Verallgemeinerung des Verzerrungssatzes auf mehrfach zusammenhängende Gebiete*, Akademie der Wissenschaften, Berlin, 1928, Part 2, pp. 228–232 and pp. 280–282; 1929, Part 2, pp. 55–62.

5. G. Szegö, *Über einige Extremalaufgaben der Potentialtheorie*, Mathematische Zeitschrift, vol. 31 (1929–1930), pp. 583–593.

6. O. D. Kellogg, (a) *Foundations of Potential Theory*, Berlin, 1929.

(b) *Singular manifolds among those of an analytic family*, this Bulletin, vol. 35 (1929), pp. 711–716.

7. E. Landau, *Über die Blochsche Konstante und zwei verwandte Weltkonstanten*, Mathematische Zeitschrift, vol. 30 (1929), pp. 608–634.

8. R. M. Robinson, *The Bloch constant for a schlicht function*, this Bulletin, vol. 41 (1935), pp. 535–540.

9. H. Grötsch, (a) *Über konforme Abbildung unendlich vielfach zusammenhängende schlichter Bereiche mit endlich vielen Haufungsrandkomponenten*, Berichte, Sächsische Akademie der Wissenschaften, Mathematische-Physikalische Klasse, vol. 81 (1929), pp. 51–86.

(b) *Über ein Variationsproblem der konformen Abbildung*, ibid., vol. 82 (1930), pp. 251–263.

10. G. C. Evans, (a) *Surfaces of minimal capacity*, Proceedings of the National Academy of Sciences, vol. 26 (1940), pp. 489–491.

(b) *Surfaces of minimum capacity*, ibid., pp. 664–667.

UNIVERSITY OF CALIFORNIA

CONDUCTION OF HEAT IN REGIONS BOUNDED BY PLANES AND CYLINDERS

J. C. JAEGER

1. **Introduction.** The Green's functions for regions bounded by surfaces of the cylindrical coordinate system are well known.[1] From them solutions may be obtained for problems in which the initial temperature is zero and the surfaces are kept at temperatures which are known functions of the coordinates; the application of Green's function in regions extending to infinity has not been completely studied and the conditions to be satisfied by functions prescribed in such regions are not known. An alternative method of solving such problems consists of using the Laplace transformation and solving the resulting subsidiary equation by separation of variables; here it is found necessary to assume that temperatures given on surfaces extending to infinity satisfy very narrow conditions, such as those of Fourier's or Weber's integral theorems. This is illustrated in §3. Problems of conduction of heat in regions bounded by cylinders and planes, some of which extend to infinity, and with constant surface (and initial) temperatures, or with a radiation boundary condition

$$\frac{\partial v}{\partial n} + h(v - v_0) = 0$$

at a surface, are of considerable importance and the constant surface temperatures do not satisfy the conditions referred to above [cf. §3]. The method given below gives a simple solution of all such problems; the results given form a complete set from which the solutions of all temperature problems in solids bounded by a cylinder and planes perpendicular to its axis, with constant surface (and initial) temperatures, can be written down. Problems involving a radiation boundary condition at some of the surfaces, and problems on the hollow cylinder, may be solved in the same way.

The method was suggested by that given in a previous note[2] which consisted of the use of a double Laplace transformation; this gives a solution under very wide conditions on the surface temperatures, roughly that they be of exponential type in the space variable. This method also allows verification that the solution obtained does satisfy

[1] Carslaw, *Conduction of Heat* (2d edition, 1921) chaps. 9 and 10; for a discussion using the Laplace transformation see Carslaw and Jaeger, Journal of the London Mathematical Society, vol. 15 (1940), p. 273.

[2] This Bulletin, vol. 46 (1940), pp. 687–693. This paper will be referred to as I.

the differential equation and boundary conditions. In I it was applied to problems on the semi-infinite strip but it may also be applied to problems involving cylindrical boundaries. The form of the solutions obtained by this process suggests a simple method of solving problems of the types referred to above. This is here applied to obtain solutions for regions bounded by surfaces of the cylindrical coordinate system with constant surface temperatures. In §2 one such problem is solved in detail to illustrate the method, and in §3 the solution is compared with that obtained by direct separation of variables. Other problems are solved briefly in §§4–7.

The method given here is formal both in the application of the Laplace transformation and in that infinite processes are interchanged without justification; in special cases it can be verified directly that the results do satisfy the differential equations and initial and boundary conditions; as remarked above this is best done by the method of I, when applicable. The use of the Laplace transformation and the method of evaluating and justifying solutions by the use of the inversion theorem followed by contour integration will be assumed known[3] and used freely without comment.

2. **Heat conduction in the semi-infinite cylinder $z>0$, $0\leqq r<a$; $r=a$ maintained at unit temperature for $z>0$ and $t>0$; $z=0$ maintained at zero for $0\leqq r<a$, $t>0$; the initial temperature zero.** We have to solve

$$(1)\qquad \frac{\partial^2 v}{\partial r^2}+\frac{1}{r}\frac{\partial v}{\partial r}+\frac{\partial^2 v}{\partial z^2}-\frac{1}{\kappa}\frac{\partial v}{\partial t}=0, \qquad 0\leqq r<a, z>0, t>0,$$

with

$$(2)\qquad v=1, \quad r=a, \quad z>0, t>0;$$

$$(3)\qquad v=0, \quad z=0, \quad 0\leqq r<a, t>0;$$

$$(4)\qquad v=0, \quad t=0, \quad z>0, 0\leqq r<a.$$

We multiply (1) and its boundary conditions (2) and (3) by e^{-pt}, $R(p)>0$, then writing

$$(5)\qquad \bar{v}(p)=\int_0^\infty e^{-pt}v(t)dt$$

for the Laplace transform of v we obtain the subsidiary equation

[3] Cf. Carslaw and Jaeger, Proceedings of the Cambridge Philosophical Society, vol. 35 (1939), p. 394, and this Bulletin, vol. 45 (1939), p. 407.

$$\frac{\partial^2 \bar{v}}{\partial r^2} + \frac{1}{r}\frac{\partial \bar{v}}{\partial r} + \frac{\partial^2 \bar{v}}{\partial z^2} - q^2 \bar{v} = 0, \quad 0 \leqq r < a, z > 0, \tag{6}$$

where

$$q^2 = p/\kappa. \tag{7}$$

This notation will be used throughout. Equation (6) is to be solved with boundary conditions

$$\bar{v} = 1/p, \qquad r = a, \qquad z > 0; \tag{8}$$

$$\bar{v} = 0, \qquad z = 0, \qquad 0 \leqq r < a. \tag{9}$$

Equation (6) is satisfied by terms of type

$$J_0(r\alpha_m)e^{-z(q^2+\alpha_m^2)^{1/2}} \tag{10}$$

where α_m, $m = 1, 2, \cdots$, are the positive roots of

$$J_0(a\alpha) = 0. \tag{11}$$

As suggested by the form of the results obtained in I, we seek a solution of (6) of type

$$\frac{1}{p}\frac{I_0(qr)}{I_0(qa)} + \sum_{m=1}^{\infty} A_m J_0(r\alpha_m)e^{-z(q^2+\alpha_m^2)^{1/2}} \tag{12}$$

in which the first term is the solution of the subsidiary equation for the infinite cylinder with unit surface temperature and the second a series of correcting terms which satisfy (6) and vanish when $r = a$, and whose coefficients A_m are to be chosen so that (12) satisfies (9). This requires

$$\frac{1}{p}\frac{I_0(qr)}{I_0(qa)} + \sum_{m=1}^{\infty} A_m J_0(r\alpha_m) = 0, \qquad 0 \leqq r < a, \tag{13}$$

that is,

$$A_m = -\frac{2\alpha_m}{ap(q^2 + \alpha_m^2)J_1(a\alpha_m)}.$$

Thus

$$\bar{v} = \frac{I_0(qr)}{pI_0(qa)} - \frac{2}{a}\sum_{m=1}^{\infty} \frac{\alpha_m}{p(q^2 + \alpha_m^2)}\frac{J_0(r\alpha_m)}{J_1(a\alpha_m)}e^{-z(\alpha_m^2+q^2)^{1/2}} \tag{14}$$

satisfies the subsidiary equation (6) and its boundary conditions (8) and (9). As remarked in §1 this result is obtained directly by the method of I.

From (14) $v(t)$ is found in the usual way[4] by the use of the inversion theorem for the Laplace transformation and we have finally

$$v(t) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} e^{\lambda t}\bar{v}(\lambda)d\lambda = 1 - \frac{2}{a}\sum_{m=1}^{\infty}\frac{J_0(r\alpha_m)e^{-z\alpha_m}}{\alpha_m J_1(a\alpha_m)} - \frac{4}{\pi a}\sum_{m=1}^{\infty}\frac{\alpha_m J_0(r\alpha_m)}{J_1(a\alpha_m)}\int_0^{\infty}\frac{e^{-\kappa t(u^2+\alpha_m^2)}\sin zu\,du}{u(u^2+\alpha_m^2)}. \tag{15}$$

This may easily be put in the more suitable form for computation

$$v = 1 - \frac{1}{a}\sum_{m=1}^{\infty}\frac{J_0(r\alpha_m)}{\alpha_m J_1(a\alpha_m)}\left\{2e^{-\kappa\alpha_m^2 t}\operatorname{erf}\frac{z}{2(\kappa t)^{1/2}} + e^{z\alpha_m}\operatorname{erfc}\left[\frac{z}{2(\kappa t)^{1/2}} + \alpha_m(\kappa t)^{1/2}\right] + e^{-z\alpha_m}\operatorname{erfc}\left[\frac{z}{2(\kappa t)^{1/2}} - \alpha_m(\kappa t)^{1/2}\right]\right\}. \tag{16}$$

3. **Comparison of the solution of §2 with that obtained formally by direct separation of variables in the subsidiary equation.** Consider the problem of §2 but with $r=a$ maintained at $f(z)$ in place of unity for $t>0$, $z>0$. The subsidiary equation (6) has to be solved with boundary condition (9) and with

$$\bar{v} = \frac{1}{p}f(z), \qquad r = a,\ z > 0. \tag{17}$$

We seek a solution composed of terms of type $I_0(q'r)\sin uz$, where

$$q' = (q^2 + u^2)^{1/2}. \tag{18}$$

Such a solution will be

$$\bar{v} = \int_0^{\infty} g(u)I_0(q'r)\sin uz\,du.$$

This satisfies (6) and (9), and (17) requires

$$\int_0^{\infty} g(u)I_0(q'a)\sin uz\,du = \frac{1}{p}f(z).$$

If we assume that $f(z)$ satisfies conditions for Fourier's integral theorem, for example, $f(z)$ satisfies Dirichlet's conditions and $\int_0^{\infty}f(z)dz$ converges absolutely, $g(u)$ can be determined and we obtain

[4] Cf. Carslaw and Jaeger, loc. cit.

$$\bar{v} = \frac{2}{\pi}\int_0^\infty \frac{I_0(rq')}{pI_0(aq')} \sin uz\, du \int_0^\infty f(z') \sin uz'\, dz', \tag{19}$$

and hence, using the inversion theorem for the Laplace transformation,

$$v = \frac{2}{\pi}\int_0^\infty \left\{\frac{I_0(ru)}{I_0(au)} - \frac{2}{a}\sum_{m=1}^\infty \frac{\alpha_m e^{-\kappa(\alpha_m^2+u^2)t} J_0(r\alpha_m)}{(\alpha_m^2+u^2)J_1(a\alpha_m)}\right\} \sin uz\, du \int_0^\infty f(z') \sin uz'\, dz', \tag{20}$$

where the α_m, $m=1, 2, \cdots$, are the positive roots of $J_0(a\alpha)=0$ and the method of solution is available only for the above restricted values of $f(z)$. However, if we invert the orders of integration in (19) convergent solutions may be obtained for functions $f(z)$ which do not satisfy the conditions above, for example (15) may be obtained in this way.

4. **The region $r>a$, $0<z<l$; $z=l$ kept at unity for $t>0$; the other surfaces at zero; the initial temperature zero.** Here the subsidiary equation is

$$\frac{\partial^2\bar{v}}{\partial r^2} + \frac{1}{r}\frac{\partial\bar{v}}{\partial r} + \frac{\partial^2\bar{v}}{\partial z^2} - q^2\bar{v} = 0, \quad r > a, 0 < z < l, \tag{21}$$

to be solved with

$$\bar{v} = 1/p, \quad z = l, \quad r > a; \tag{22}$$

$$\bar{v} = 0, \quad z = 0, \quad r > a; \tag{23}$$

$$\bar{v} = 0, \quad r = a, \quad 0 < z < l. \tag{24}$$

The solution of the subsidiary equation for the infinite flat plate $0<z<l$ with boundary conditions (22) and (23) is $(\sinh qz)/(p \sinh ql)$, so we choose for the solution of (21)

$$\bar{v} = \frac{\sinh qz}{p \sinh ql} + \sum_{n=1}^\infty A_n \sin\frac{n\pi z}{l} K_0(rq_n) \tag{25}$$

where

$$q_n = (q^2 + n^2\pi^2/l^2)^{1/2}. \tag{26}$$

This satisfies (21), (22), and (23), and (24) requires

$$\frac{\sinh qz}{p \sinh ql} + \sum_{n=1}^\infty A_n \sin\frac{n\pi z}{l} K_0(aq_n) = 0$$

that is,

$$A_n = \frac{2n\pi(-)^n}{pl^2q_n^2K_0(aq_n)}.$$

Thus

$$\bar{v} = \frac{\sinh qz}{p\sinh ql} + \frac{2\pi}{pl^2}\sum_{n=1}^{\infty}\frac{n(-)^nK_0(rq_n)}{q_n^2K_0(aq_n)}\sin\frac{n\pi z}{l}. \tag{27}$$

And, using the inversion theorem for the Laplace transformation,

$$\begin{aligned} v = {} & \frac{z}{l} + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-)^nK_0(n\pi r/l)}{nK_0(n\pi a/l)}\sin\frac{n\pi z}{l} \\ & - \frac{4}{l^2}\sum_{n=1}^{\infty}\left\{n(-)^ne^{-\kappa n^2\pi^2t/l^2}\sin\frac{n\pi z}{l}\right. \\ & \left.\cdot\int_0^{\infty}\frac{e^{-\kappa u^2t}C_0(ur, ua)du}{u(u^2+n^2\pi^2/l^2)[J_0^2(ua)+Y_0^2(ua)]}\right\}, \end{aligned} \tag{28}$$

where

$$C_0(x, y) = J_0(x)Y_0(y) - Y_0(x)J_0(y). \tag{29}$$

5. **The region $r>a$, $z>0$; $z=0$ kept at unity for $t>0$, $r>a$; $r=a$ kept at zero for $z>0$, $t>0$; the initial temperature zero.** Here

$$\bar{v} = \frac{1}{p}e^{-qz} - \frac{2}{\pi p}\int_0^{\infty}\frac{u\sin uz}{u^2+q^2}\frac{K_0(rq')}{K_0(aq')}du, \tag{30}$$

where q and q' are defined in (7) and (18) respectively, satisfies the subsidiary equation and its boundary condition at $z=0$. It also satisfies the boundary condition at $r=a$ since

$$\frac{2}{\pi}\int_0^{\infty}\frac{u\sin uz}{u^2+q^2}du = e^{-qz}.$$

Using the inversion theorem we obtain from (30)

$$\begin{aligned} v = {} & 1 - \frac{2}{\pi}\int_0^{\infty}\frac{K_0(ur)\sin uz}{uK_0(ua)}du + \frac{4}{\pi^2}\int_0^{\infty}e^{-\kappa u^2t}u\sin uz\,du \\ & \cdot\int_0^{\infty}\frac{e^{-\kappa u'^2t}C_0(ru', au')du'}{u'(u'^2+u^2)[J_0^2(au')+Y_0^2(au')]}. \end{aligned} \tag{31}$$

6. **The region $r>a$, $z>0$; $r=a$ kept at unity for $z>0$, $t>0$; $z=0$ kept at zero for $r>a$, $t>0$; the initial temperature zero.** Here

$$(32)\qquad \bar{v} = \frac{1}{p}\frac{K_0(qr)}{K_0(qa)} + \frac{2}{\pi p}\int_0^\infty e^{-zq'}\frac{C_0(ur, ua)udu}{(q^2+u^2)[J_0^2(ua)+Y_0^2(ua)]}$$

satisfies the subsidiary equation and its boundary condition at $r=a$. It also satisfies the boundary condition at $z=0$, since, by Weber's integral theorem,

$$\frac{K_0(qr)}{K_0(qa)} + \frac{2}{\pi}\int_0^\infty \frac{C_0(ur, ua)udu}{(q^2+u^2)[J_0^2(ua)+Y_0^2(ua)]} = 0.$$

It follows that

$$(33)\qquad \begin{aligned} v = 1 &+ \frac{2}{\pi}\int_0^\infty \frac{e^{-u^2}C_0(ru, au)du}{u[J_0^2(au)+Y_0^2(au)]} \\ &+ \frac{4}{\pi^2}\int_0^\infty \frac{e^{-\kappa u^2 t}C_0(ru, au)udu}{J_0^2(au)+Y_0^2(au)}\int_0^\infty \frac{e^{-\kappa u'^2 t}\sin u'z\,du'}{u'(u^2+u'^2)}. \end{aligned}$$

7. **The finite cylinder and similar problems.** In this section four results are given for completeness which are obtained by the method of §2; these problems may also be solved by the method of §3, and in all cases the transient parts of the solutions are identical, but the steady state parts are obtained in a different form.

(i) *The region* $0\leqq r<a$, $z>0$; $z=0$ *kept at unity for* $t>0$, $0\leqq r<a$; $r=a$ *kept at zero for* $z>0$, $t>0$; *the initial temperature zero*:

$$(34)\qquad \bar{v} = \frac{1}{p}e^{-qz} - \frac{2}{\pi p}\int_0^\infty \frac{u\sin uz}{q^2+u^2}\frac{I_0(rq')}{I_0(aq')}du,$$

where q and q' are defined in (7) and (18);

$$(35)\qquad \begin{aligned} v = 1 &- \frac{2}{\pi}\int_0^\infty \frac{I_0(ur)\sin uz\,du}{uI_0(ua)} \\ &- \frac{4}{\pi a}\sum_{m=1}^\infty \frac{e^{-\kappa a_m^2 t}J_0(r\alpha_m)}{\alpha_m J_1(a\alpha_m)}\int_0^\infty \frac{e^{-\kappa u^2 t}u\sin uz\,du}{(\alpha_m^2+u^2)}, \end{aligned}$$

where the α_m are the positive roots of $J_0(a\alpha)=0$.

(ii) *The region* $r>a$, $0<z<l$; $r=a$ *kept at unity for* $t>0$, $0<z<l$; *the other surfaces kept at zero for* $t>0$; *the initial temperature zero*:

$$(36)\qquad \bar{v} = \frac{K_0(qr)}{pK_0(qa)} + \int_0^\infty \frac{2\cosh q'(\frac{1}{2}l-z)C_0(ru, au)udu}{\pi pq'^2\cosh\frac{1}{2}q'l[J_0^2(ua)+Y_0^2(ua)]},$$

$$v = 1 + \frac{2}{\pi}\int_0^\infty \frac{\cosh u(\frac{1}{2}l - z)C_0(ru, au)du}{u \cosh \frac{1}{2}lu[J_0^2(au) + Y_0^2(au)]}$$

$$+ \frac{8}{\pi^2}\sum_{n=0}^{\infty}\left\{e^{-\kappa(2n+1)^2\pi^2 t/l^2}\frac{\sin (2n+1)\pi z/l}{(2n+1)}\right.$$

$$\left.\cdot\int_0^\infty \frac{e^{-\kappa u^2 t}C_0(ru, au)udu}{[u^2 + (2n+1)^2\pi^2/l^2][J_0^2(au) + Y_0^2(au)]}\right\}. \tag{37}$$

(iii) *The finite cylinder* $0 \leqq r < a$, $0 < z < l$; $r = a$ *kept at unity and the other surfaces at zero for* $t > 0$; *the initial temperature zero*:

$$\bar{v} = \frac{I_0(qr)}{pI_0(qa)} - \frac{2}{a}\sum_{m=1}^{\infty}\frac{\alpha_m J_0(r\alpha_m)}{pq_m^2 J_1(a\alpha_m)}\frac{\cosh (\frac{1}{2}l - z)q_m}{\cosh \frac{1}{2}lq_m}, \tag{38}$$

where α_m, $m = 1, 2, \cdots$, are the positive roots of $J_0(a\alpha) = 0$ and $q_m = (q^2 + \alpha_m^2)^{1/2}$;

$$v = 1 - \frac{2}{a}\sum_{m=1}^{\infty}\frac{J_0(r\alpha_m)}{\alpha_m J_1(a\alpha_m)}\frac{\cosh (\frac{1}{2}l - z)\alpha_m}{\cosh \frac{1}{2}l\alpha_m}$$

$$- \frac{8}{a\pi}\sum_{m=1}^{\infty}\left\{\frac{\alpha_m J_0(r\alpha_m)}{J_1(a\alpha_m)}e^{-\kappa\alpha_m^2 t}\right.$$

$$\left.\cdot\sum_{n=0}^{\infty}\frac{e^{-\kappa(2n+1)^2\pi^2 t/l^2}\sin (2n+1)\pi z/l}{(2n+1)[\alpha_m^2 + (2n+1)^2\pi^2/l^2]}\right\}. \tag{39}$$

(iv) *The finite cylinder* $0 \leqq r < a$, $0 < z < l$; $z = l$ *kept at unity and the other surfaces at zero for* $t > 0$; *the initial temperature zero*:

$$\bar{v} = \frac{\sinh qz}{p \sinh ql} + \frac{2\pi}{pl^2}\sum_{n=1}^{\infty}\frac{n(-)^n I_0(rq_n)}{q_n^2 I_0(aq_n)}\sin\frac{n\pi z}{l}, \tag{40}$$

where $q_n = (q^2 + n^2\pi^2/l^2)^{1/2}$;

$$v = \frac{z}{l} + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-)^n I_0(n\pi r/l)}{nI_0(n\pi a/l)}\sin\frac{n\pi z}{l}$$

$$+ \frac{4\pi}{a}\sum_{n=1}^{\infty}n(-)^n \sin\frac{n\pi z}{l}\sum_{m=1}^{\infty}\frac{J_0(r\alpha_m)e^{-\kappa t(\alpha_m^2 + n^2\pi^2/l^2)}}{\alpha_m(l^2\alpha_m^2 + n^2\pi^2)J_1(a\alpha_m)}, \tag{41}$$

where α_m, $m = 1, 2, \cdots$, are the positive roots of $J_0(a\alpha) = 0$.

THE UNIVERSITY OF TASMANIA,
HOBART, TASMANIA

ON A THEOREM BY J. L. WALSH CONCERNING THE MODULI OF ROOTS OF ALGEBRAIC EQUATIONS

ALEXANDER OSTROWSKI

In 1881 A. E. Pellet published[1] the following very useful theorem:

If the polynomial

$$F(z) \equiv |a_0| + |a_1|z + |a_2|z^2 + \cdots + |a_{k-1}|z^{k-1} - |a_k|z^k + |a_{k+1}|z^{k+1} + \cdots + |a_n|z^n, \tag{1}$$

$$0 < k < n,\ a_0 a_n \neq 0,$$

has two positive roots x_1 and x_2 $(x_1 < x_2)$, then the polynomial

$$f(z) \equiv a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n \tag{2}$$

has no roots in the annulus $x_1 < |z| < x_2$ and precisely k roots in the circle $|z| \leqq x_1$.

While Pellet's proof for his theorem utilizes the theorem of Rouché, J. L. Walsh published in 1924[2] another more direct proof and established in the same memoir a sort of converse of Pellet's theorem. To formulate his result, consider the set $\mathfrak{F}$ of all polynomials

$$f(z) \equiv a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n \tag{3}$$

which correspond to given moduli of coefficients. All polynomials of $\mathfrak{F}$ are obtained from one of them, $f(z)$, if the factors $\epsilon_0, \epsilon_1, \cdots, \epsilon_n$ in the expression

$$\epsilon_0 a_0 + \epsilon_1 a_1 z + \epsilon_2 a_2 z^2 + \cdots + \epsilon_n a_n z^n \tag{4}$$

assume independently all values of modulus 1. Let $\mathfrak{M}$ be the set of roots of all polynomials in $\mathfrak{F}$. It is immediately seen that if $\mathfrak{M}$ contains a number α it also contains all numbers with the modulus $|\alpha|$.

It was proved by Cauchy that all roots of (4) lie on or within the circle $|z| = y_1$, where y_1 is the single positive root of the polynomial

$$|a_0| + |a_1|z + \cdots + |a_{n-1}|z^{n-1} - |a_n|z^n$$

and that all roots of (4) lie on or are exterior to the circle $|z| = y_2$, where y_2 is the single positive root of the polynomial

---

[1] A. E. Pellet: *Sur un mode de séparation des racines des équations et la formule de Lagrange*, Bulletin des Sciences Mathématiques, (2), vol. 5 (1881), pp. 393–395.

[2] J. L. Walsh: *On Pellet's theorem concerning the roots of a polynomial*, Annals of Mathematics, vol. 26 (1924), pp. 59–64.

$$|a_0| - |a_1| z - \cdots - |a_n| z^n.$$

Since the numbers y_1 and y_2 obviously belong to $\mathfrak{M}$, it follows that $\mathfrak{M}$ lies in the closed annulus $y_2 \leqq |z| \leqq y_1$.

Suppose now that $a>0$ is not contained in $\mathfrak{M}$. Then, if C is the circle $|z|=a$, all polynomials of $\mathfrak{F}$ contain the same number k of roots within C. Indeed, if we vary continuously the factors ϵ_ν in the representation (4), the roots of (4) vary also continuously and their number within C remains the same since none of them is able to pass C.

Now the theorem of Walsh runs as follows:

Any positive a that is not contained in $\mathfrak{M}$, and for which the number of roots of (4) *within the circle $|z|=a$ is precisely k $(0<k<n)$, is contained between the two positive roots of the polynomial* (1), *that is to say*:

$$|a_k| a^k > (|a_0| + \cdots + |a_{k-1}| a^{k-1}) + (|a_{k+1}| a^{k+1} + \cdots + |a_n| a^n). \tag{5}$$

As the proof given by Walsh of this important result is not complete,[3] we give in what follows another proof proceeding on different lines.

Suppose that, contrary to (5),

$$|a_k| a^k \leqq \sum_{\nu=0,\nu\neq k}^{n} |a_\nu| a^\nu; \tag{6}$$

then, as a is not contained in $\mathfrak{M}$, we have even

$$|a_k| a^k < \sum_{\nu=0,\nu\neq k}^{n} |a_\nu| a^\nu. \tag{7}$$

On the other hand it follows from our hypothesis that if ϵ_ν run independently through all constants of modulus 1,

$$\sum_{\nu=0}^{n} \epsilon_\nu a_\nu a^\nu \neq 0, \tag{8}$$

and therefore

[3] Walsh allows in his proof the roots of $f(z)$, which are in absolute value less than a, to vary continuously and monotonically (in absolute value) and to approach 0. But during this variation the polynomial $f(z)$ does not necessarily remain in the set $\mathfrak{F}$ and the corresponding sets $\mathfrak{F}_0$ for the polynomials thus obtained could very well contain a in the set of the roots, so that the expression (1) need not remain positive for $z=a$, as is assumed in Walsh's proof.

$$|a_k|a^k \neq \left| \sum_{\nu=0,\nu\neq k} \epsilon_\nu a_\nu a^\nu \right|. \tag{9}$$

But from (7) and (9) it follows that for all values of ϵ_ν in question

$$|a_k|a^k < \left| \sum_{\nu=0,\nu\neq k}^{n} \epsilon_\nu a_\nu a^\nu \right|. \tag{10}$$

For, if we have for a particular set of ϵ_ν-values, ϵ_ν^0:

$$|a_k|a^k > \left| \sum_{\nu=0,\nu\neq k}^{n} \epsilon_\nu^0 a_\nu a^\nu \right|, \tag{11}$$

we see from (7) and (11), that the right-hand side of (10) becomes equal to $|a_k|a^k$ for some other set of ϵ_ν, contrary to (9).

Now it follows from (10) that the polynomials

$$f^*(z) = \sum_{\nu=0,\nu\neq k}^{n} \epsilon_\nu a_\nu z^\nu \tag{12}$$

have no roots at a and therefore no roots on the circle C. On the other hand, we have obviously by (10) everywhere on C: $|f^*(z)| > |a_k z^k|$. Hence, by the theorem of Rouché, since $f^*(z) - a_k z^k$ has exactly k roots inside C, the same is true for any polynomial $f^*(z)$.

The result arrived at may be announced in the following form:

If $\epsilon_0, \cdots, \epsilon_{k-1}, \epsilon_{k+1}, \cdots, \epsilon_n$ run independently through all values of modulus 1, let

$$\phi(z) = \sum_{\nu=0}^{k-1} \epsilon_\nu a_\nu z^\nu, \qquad \psi(z) = \sum_{\nu=k+1}^{n} \epsilon_\nu a_\nu z^\nu;$$

then the difference $\phi(z) - \psi(z)$ does not vanish on C and has exactly k roots inside C.

It follows, that in particular

$$\left| \sum_{\nu=0}^{k-1} \epsilon_\nu a_\nu a^\nu \right| \neq \left| \sum_{\nu=k+1}^{n} \epsilon_\nu a_\nu a^\nu \right|.$$

But then it is impossible that we have simultaneously for one particular set ϵ_ν' of ϵ_ν

$$\left| \sum_{\nu=0}^{k-1} \epsilon_\nu' a_\nu a^\nu \right| > \left| \sum_{\nu=k+1}^{n} \epsilon_\nu' a_\nu a^\nu \right|$$

and for another, ϵ_ν'',

$$\left|\sum_{\nu=0}^{k-1}{}'' a_\nu a^\nu\right| < \left|\sum_{\nu=k+1}^{n}{}'' a_\nu a^\nu\right|.$$

Hence only two cases are possible:

A. *We have always*

$$\left|\sum_{\nu=0}^{k-1}\epsilon_\nu a_\nu a^\nu\right| > \left|\sum_{\nu=k+1}^{n}\epsilon_\nu a_\nu a^\nu\right|,$$

and therefore everywhere on C: $|\phi(z)| > |\psi(z)|$.

But then, by the theorem of Rouché, $\phi(z)$ has inside C the same number of roots as $\phi(z)-\psi(z)$, that is k, and this is impossible, $\phi(z)$ being of degree $k-1$.

B. *We have always*

$$\left|\sum_{\nu=0}^{k-1}\epsilon_\nu a_\nu a^\nu\right| < \left|\sum_{\nu=k+1}^{n}\epsilon_\nu a_\nu a^\nu\right|,$$

and therefore everywhere on C: $|\phi(z)| < |\psi(z)|$.

But then $\psi(z)$ would have inside C exactly k roots, while $\psi(z)$ vanishes at $z=0$ with the multiplicity $k+1$ at least.

We see that (7) and (6) are impossible and the proof of (5) is completed.

The theorems of Pellet and Walsh dealt with in the preceding paragraphs allow us to describe immediately the set $\mathfrak{M}$.

Consider the $n+1$ equations:

$$|a_n| z^n - \sum_{\nu=0}^{n-1} |a_\nu| z^\nu = 0, \tag{13}$$

$$\sum_{\nu=k+1}^{n} |a_\nu| z^\nu - |a_k| z^k + \sum_{\nu=0}^{k-1} |a_\nu| z^\nu = 0, \qquad k = 1, \cdots, n-1, \tag{14}$$

$$\sum_{\nu=1}^{n} |a_\nu| z^\nu - |a_0| = 0. \tag{15}$$

If, as we will assume, $a_0 a_n \neq 0$, each of the equations (13), (15) possesses one positive root ρ resp. ρ_0, and we have $\rho_0 \leqq \rho$ since ρ is the exact upper and ρ_0 the exact lower limit for the moduli of $\mathfrak{M}$. As to the equations (14), every one of them possesses either two positive roots or exactly one double positive root or no positive roots at all. Strike out the equations (14) corresponding to the two last cases; each of the remaining equations (14) possesses two positive roots, x_k, x_k'

$(x_k < x_k')$. Then we obtain $\mathfrak{M}$ by removing from the plane of z the two circular domains $|z| < \rho_0$ and $|z| > \rho$ and all annuli $x_k < |z| < x_k'$, corresponding to the equations (14) with two different positive roots.

It is then a consequence of Walsh's theorem that the different (open) intervals (x_k, x_k') have no points in common and lie in the interval (ρ_0, ρ); moreover, if for two of these intervals (x_k, x_k'), (x_m, x_m') we have $k < m$, then we have certainly $x_k' \leqq x_m$, and it is easily seen that we have even $x_k' < x_m$.

As Walsh remarks, his proof of Pellet's theorem remains valid also in the case of a power-series and of its roots inside the circle of convergence. It is hardly necessary to remark that our proof of Walsh's theorem also applies *mutatis mutandis* to a power series, if we only consider its roots within the circle of convergence.

UNIVERSITY OF BASEL

SOME EXCEPTIONAL VALUES OF THE LIMIT OF THE RATIO OF ARC TO CHORD

RICHARD COHN

It was observed by E. Kasner[1] that in the complex euclidean plane the limiting value of the ratio of the arc of a curve to its chord, while one end point of the arc is fixed, and the other approaches it along the curve, is not always unity; but assumes for analytic curves tangent to a minimal line, a sequence of real values, $.94 \cdots, .86 \cdots, .80, \cdots$. These values are functions of the order of contact only, and approach zero as the latter increases. In this note we shall describe two similar situations which occur in real spaces.

The problem in the case of the K plane[2] has been worked out in Professor Kasner's Seminar in Geometry.[3] In this plane the length of the curve $y = f(x)$ passing between points of abscissae x_1, x_2, in that order, is given by

$$l = \int_{x_1}^{x_2} \left(\frac{dy}{dx}\right)^{-1} dx. \tag{1}$$

[1] E. Kasner, *The ratio of the arc to the chord of an analytic curve need not approach unity*, this Bulletin, vol. 21 (1914), pp. 524–531. Similar questions for three dimensions are discussed in E. Kasner, *Complex geometry and relativity, theory of the "rac" curvature*, Proceedings of the National Academy of Sciences, 1932, p. 267.

[2] Kasner, *Trihornometry, a new chapter of conformal geometry*, Proceedings of the National Academy of Sciences, vol. 23, p. 346.

[3] R. Coleman, S. Jablon and D. Mittleman obtained the results for the K plane given below.

The distance from the point (x_1, y_1) to the point (x_2, y_2) is defined as the length of the straight line passing from the first of these points to the second and is therefore given by the equation

$$c = \frac{(x_2 - x_1)^2}{y_2 - y_1}. \tag{2}$$

We see that lines parallel to the x-axis have infinite length while those parallel to the y-axis are the minimal lines or lines of zero length. We shall now investigate the ratio of the length of an analytic arc to the distance between its end points, both length and distance being used in the sense just defined.

The most interesting cases will be those of curves tangent to a minimal line. We will obviously lose nothing by choosing the origin for the point of contact. We will write the equation of our curve in the form $x = a_n y^n + a_{n+1} y^{n+1} + \cdots$, $a_n \neq 0$. Evidently this includes the normal as well as the exceptional situations; for when $n = 1$ the curve cuts the minimal line, for greater values of n it is tangent to it.

We may write for the length of the curve from the origin to the point (X, Y) the equation $l = \int_0^Y (dx/dy)^2 dy$. This is obtained easily from the expression (1) for the length. Since $dx/dy = na_n y^{n-1} + \cdots$, this becomes

$$\begin{aligned} l &= \int_0^Y (na_n y^{n-1} + \cdots)^2 dy = \int_0^Y (n^2 a_n^2 y^{2n-2} + \cdots) dy, \\ l &= \frac{n^2}{2n-1} a_n^2 Y^{2n-1} + \cdots. \end{aligned} \tag{3}$$

The distance from the origin to the point (X, Y) is seen to be

$$c = \frac{X^2}{Y} = \frac{(a_n Y^n + \cdots)^2}{Y} = a_n^2 Y^{2n-1} + \cdots. \tag{4}$$

The first terms of these expansions do not vanish, so that as Y approaches zero, l/c approaches their quotient $n^2/(2n-1)$ as limit. *For $n = 1$ this is unity, but for curves tangent to the minimal line this limit assumes a sequence of values* 4/3, 9/5, 25/4, $\cdots$, *increasing towards infinity with the order of contact.*

There remains the case of curves tangent to a line of infinite length. These must be written in the form $y = a_n x^n + \cdots$, $a_n \neq 0$, $n > 1$. A calculation similar to the preceding one shows that such curves have infinite arc length measured from the point of tangency to any other point, while the corresponding chords are finite.

In the affine differential geometry of the plane which has been developed by Blaschke[4] and others we work with elements consisting of a point and a slope. For a curve to pass from the element consisting of the point (x_1, y_1) and the slope y_1' to the element consisting of the point (x_2, y_2) and the slope y_2' it must pass through each point and have at each the corresponding slope. If $y=f(x)$ is the equation of such a curve, the affine length of the arc between these elements is defined by

$$l = \int_{x_1}^{x_2} \left(\frac{d^2y}{dx^2}\right)^{1/3} dx. \tag{5}$$

This integral is invariant under an area-preserving affine transformation. With the given end-conditions it becomes stationary when the arc is a parabola. Its value in this case is a relative maximum and is called the affine distance between the elements.

I have carried over the problem to this case by considering the limit of the ratio of the affine length of an arc to the affine distance between the elements tangent to the arc at its end points, as one end point is fixed and the other approaches it along the curve.

It will be observed that all straight lines have zero affine length, and it is to be expected, by analogy with the preceding cases, that unusual values of the limit will occur for those curves and only those which have high order of contact with a straight line. That this is indeed the case is shown in the following calculations.

By means of a suitable translation and rotation, which cannot affect the quantities under investigation, we write any analytic curve in the form $y=a_nx^n+a_{n+1}x^{n+1}+\cdots$, $a_n\neq 0$, $n\geqq 2$. The affine length of this curve from the element at the origin with slope zero to the element tangent to the curve at the point (X, Y) is found by means of (5) to be given by

$$l = \int_0^X \left(\frac{d^2y}{dx^2}\right)^{1/3} dx = \int_0^X (n(n-1)a_nx^{n-2}+\cdots)^{1/3}dx,$$

$$l = \int_0^X ([n(n-1)a_n]^{1/3}x^{(n-2)/3}+\cdots)dx, \tag{6}$$

$$l = \frac{[n(n-1)a_n]^{1/3}}{(n+1)/3} X^{(n+1)/3}+\cdots.$$

[4] W. Blaschke, *Differentialgeometrie*, vol. 2, chap. 1. (5) and (7) below are easily obtained from the equations given by Blaschke in vector notation.

The general formula for the affine distance between the elements consisting of the points (x_1, y_1), (x_2, y_2) with the slopes y_1' and y_2' respectively is

$$(7)\quad c = \left[\frac{4[(y_2 - y_1) - y_1'(x_2 - x_1)][y_2'(x_2 - x_1) - (y_2 - y_1)]}{y_2' - y_1'}\right]^{1/3}.$$

In the case under consideration this becomes

$$c = \left[\frac{4Y}{Y'}(XY' - Y)\right]^{1/3}, \qquad Y' = \frac{dy}{dx}\Big]_{x=X} = na_nX^{n-1} + \cdots,$$

$$(8)\quad c = \left[\frac{(4a_nX^n + \cdots)([na_nX^n + \cdots] - [a_nX^n + \cdots])}{na_nX^{n-1} + \cdots}\right]^{1/3},$$

$$c = \left[\frac{4(n-1)a_n}{n}\right]^{1/3} X^{(n+1)/3} + \cdots.$$

We observe that the first terms of the expansions for l and c do not vanish. Consequently as X approaches zero the ratio of these quantities approaches

$$\frac{3}{n+1}\left[\frac{n^2}{4}\right]^{1/3}.$$

Thus the limiting value of the ratio of affine arc to affine chord will be unity for the case $n=2$, but for curves with a higher order of contact with a straight line it will assume a sequence of values .98 $\cdots$, .95 $\cdots$, .92 $\cdots$, $\cdots$, dependent only on the order of contact and decreasing towards zero as the latter increases.

We have thus seen that there is a close agreement in the behavior of the limit of the ratio of the arc and chord of an analytic curve in the complex euclidean, the Kasner and the affine plane. *In each case this quantity is unity for most curves, but may also have one of a denumerable set of other values. These values occur for curves with higher than the least possible order of contact with a minimal line, and depend only on that order of contact.*

COLUMBIA UNIVERSITY

A NOTE ON FUNCTIONS OF EXPONENTIAL TYPE[1]

R. P. BOAS, JR.

An entire function $f(z)$ is said to be of exponential type at most T if

$$\limsup_{n\to\infty} |f^{(n)}(z)|^{1/n} \leqq T \tag{1}$$

for some z (and hence for every z, uniformly for z in any bounded set). An equivalent condition[2] is that for each positive ϵ

$$|f(z)| < e^{(T+\epsilon)|z|}$$

for all sufficiently large $|z|$. The following three theorems were proved respectively by D. V. Widder [4], I. J. Schoenberg [2], and H. Poritsky [1] and J. M. Whittaker [3].

THEOREM 1. (Widder.) *If a real function $f(z)$, of class C^∞ in $0 \leqq x \leqq 1$, satisfies the condition*

$$(-1)^n f^{(2n)}(x) \geqq 0, \qquad 0 \leqq x \leqq 1;\ n = 0, 1, 2, \cdots, \tag{2}$$

then $f(x)$ coincides on $(0, 1)$ with an entire function of exponential type at most π.

THEOREM 2. (Schoenberg.) *If $f(z)$ is an entire function of exponential type at most T, and if*

$$f^{(2n)}(0) = f^{(2n)}(1) = 0, \qquad n = 0, 1, 2, \cdots, \tag{3}$$

then $f(z)$ is a sine polynomial of order at most T/π:

$$f(z) = \sum_{k=0}^{N} a_k \sin k\pi z, \qquad N \leqq T/\pi.$$

Let $\Lambda_n(z)$ be the polynomial of degree $2n+1$ determined by the relations

$$\Lambda_0(z) = z; \qquad \Lambda_n(0) = \Lambda_n(1) = 0, \qquad n \geqq 1;$$
$$\Lambda_n''(z) = \Lambda_{n-1}(z), \qquad n \geqq 1.$$

THEOREM 3. (Poritsky-Whittaker.) *If $f(z)$ is an entire function of exponential type at most T, $T < \pi$, then $f(z)$ can be represented in the form*

---

[1] Presented to the Society, February 22, 1941.

[2] G. Valiron, *Lectures on the General Theory of Integral Functions*, Toulouse, 1923, p. 41.

$$f(z) = \sum_{n=0}^{\infty} f^{(2n)}(1)\Lambda_n(z) - \sum_{n=0}^{\infty} f^{(2n)}(0)\Lambda_n(z-1), \tag{4}$$

where the series converge uniformly in any bounded region.

An expansion of the form (4) is called a Lidstone series.

The chief purpose of this note is to give a short proof of Theorem 1 which I obtained some time after Professor Widder communicated the theorem to me.[3] I also give simple proofs of Theorems 2 and 3, suggested by the proof of Theorem 1.

LEMMA 1. *If k and n are positive integers, and $f(x)$ is of class C^{2n} in $0 \leqq x \leqq 1$, then*

$$\int_0^1 f(x) \sin k\pi x\, dx = \sum_{m=0}^{n-1} \frac{(-1)^m}{(k\pi)^{2m+1}} \left\{ f^{(2m)}(0) - (-1)^k f^{(2m)}(1) \right\} + \frac{(-1)^n}{(k\pi)^{2n}} \int_0^1 f^{(2n)}(x) \sin k\pi x\, dx. \tag{5}$$

This is proved by integrating the left-hand side repeatedly by parts.

LEMMA 2. *If $g(x)$ is non-negative and concave in $a \leqq x \leqq b$, and*

$$\int_a^b g(x)dx \leqq A,$$

then

$$g(x) \leqq \frac{2A}{b-a}, \qquad a \leqq x \leqq b. \tag{6}$$

Let $g(x)$ take its maximum, G, at $x = x_0$. Since $g(x)$ is concave, the graph of $g(x)$ is above the broken line connecting the points $(a, 0)$ and $(b, 0)$ to (x_0, G). The area under the broken line is $\frac{1}{2}G(b-a)$. Hence $\frac{1}{2}G(b-a) \leqq A$, and (6) follows.

PROOF OF THEOREM 1. Take $k=1$ in Lemma 1. By hypothesis the terms of the sum on the right of (5) are all non-negative. Hence, for any positive integer n,

$$B = \int_0^1 f(x) \sin \pi x\, dx \geqq \frac{1}{\pi^{2n}} \int_0^1 (-1)^n f^{(2n)}(x) \sin \pi x\, dx \geqq 0,$$

$$\int_\delta^{1-\delta} (-1)^n f^{(2n)}(x) \sin \pi x\, dx \leqq B\pi^{2n}, \qquad 0 < \delta < \tfrac{1}{2},$$

[3] (Added in proof.) Essentially the same proof was found independently by Professor Schoenberg.

$$\int_\delta^{1-\delta} (-1)^n f^{(2n)}(x)dx \leqq \frac{B\pi^{2n}}{\sin \pi\delta}.$$

Since $(-1)^n f^{(2n+2)}(x) \leqq 0$, $(-1)^n f^{(2n)}(x)$ is non-negative and concave in $(\delta, 1-\delta)$. By Lemma 2,

$$0 \leqq (-1)^n f^{(2n)}(x) \leqq \frac{2B\pi^{2n}}{(1-2\delta)\sin \pi\delta}, \qquad \delta \leqq x \leqq 1-\delta. \tag{7}$$

Let $h = \frac{1}{2} - \delta$. For any x in $(\delta, 1-\delta)$, Taylor's theorem with remainder of order 2, applied to $f^{(2n)}(x)$, leads to the relation

$$f^{(2n+1)}(x) = \pm \frac{1}{h}\left[f^{(2n)}(x \pm h) - f^{(2n)}(x)\right] \mp \tfrac{1}{2}hf^{(2n+2)}(x \pm \theta h), \qquad 0 < \theta < 1,$$

where the upper signs or the lower signs are taken according as $x+h$ or $x-h$ is in $(\delta, 1-\delta)$. Using (7), written both for n and $n+1$, we obtain

$$\left| f^{(2n+1)}(x) \right| \leqq \left(\frac{2}{h} + \frac{\pi^2 h}{2}\right)\frac{2B\pi^{2n}}{(1-2\delta)\sin \pi\delta} = C(\delta)B\pi^{2n+1},$$

where $C(\delta)$ depends only on δ. Thus

$$\limsup_{n\to\infty} \left| f^{(n)}(x) \right|^{1/n} \leqq \pi, \tag{8}$$

uniformly in any interval $\delta < x < 1-\delta$, where $0 < \delta < \frac{1}{2}$. The relation (8) implies first that the Taylor series of $f(x)$ about any point in $(0, 1)$ converges to an entire function coinciding with $f(x)$ in $(0, 1)$; (8) then shows that this function is of exponential type at most π.

Proof of Theorem 2. Since $f(x)$ is of class C^∞, it is represented in $(0, 1)$ by its Fourier sine series:

$$f(x) = \sum_{k=0}^{\infty} a_k \sin k\pi x, \qquad 0 < x < 1, \tag{9}$$

where

$$a_k = 2\int_0^1 f(t) \sin k\pi t\, dt.$$

Since (3) is true, Lemma 1 shows that

$$a_k = \frac{2(-1)^n}{(k\pi)^{2n}} \int_0^1 f^{(2n)}(t) \sin k\pi t\, dt$$

for every positive integer n. Hence

$$|a_k| \leqq \frac{2}{(k\pi)^{2n}} \max_{0\leqq x\leqq 1} |f^{(2n)}(x)| \tag{10}$$

If $k\pi > T$, for large n we have

$$\max_{0\leqq x\leqq 1} |f^{(2n)}(x)| \leqq S^{2n}, \qquad T < S < k\pi,$$

since $f(x)$ is of exponential type at most T. From (10) it then follows that $a_k = 0$ if $k\pi > T$. Thus all terms of (9) with $k > T/\pi$ vanish, and Theorem 2 is proved.

PROOF OF THEOREM 3. The function $f(x)$ is represented in (0, 1) by the Fourier series (9), and $\frac{1}{2}a_k$ is just the integral on the left of (5). Hence, for every positive integer n,

$$\begin{aligned} f(x) = &\sum_{m=0}^{n-1} f^{(2m)}(1) \sum_{k=1}^{\infty} \frac{2(-1)^{k+m+1} \sin k\pi x}{(k\pi)^{2m+1}} \\ &- \sum_{m=0}^{n-1} f^{(2m)}(0) \sum_{k=1}^{\infty} \frac{2(-1)^{m+1} \sin k\pi x}{(k\pi)^{2m+1}} + \tfrac{1}{2} R_n(x), \end{aligned} \tag{11}$$

where

$$R_n(x) = (-1)^n \int_0^1 f^{(2n)}(t) \sum_{k=1}^{\infty} \frac{\sin k\pi x \sin k\pi t}{(k\pi)^{2n}}\, dt.$$

The infinite series appearing in (11) are the Fourier sine series of $\Lambda_m(x)$ and $\Lambda_m(1-x)$, as given by Whittaker;[4] they can easily be checked by successive integrations of the well known Fourier sine series of $\Lambda_0(x) = x$. Relation (11) becomes

$$f(x) = \sum_{m=0}^{n-1} f^{(2m)}(1)\Lambda_m(x) - \sum_{m=0}^{n-1} f^{(2m)}(0)\Lambda_m(x-1) + R_n(x).$$

We have thus obtained a "Lidstone series with remainder," with an expression for the remainder as a real integral equivalent to that given by Widder [4].

We have

$$|R_n(x)| \leqq \frac{1}{\pi^{2n}} \sum_{k=1}^{\infty} \frac{1}{k^{2n}} \int_0^1 |f^{(2n)}(t)|\, dt,$$

and this approaches zero as $n \to \infty$, uniformly in (0, 1), if $f(z)$ satisfies (1) with $T < \pi$. Hence under the hypothesis of Theorem 3 the series in (4) converges to $f(x)$ uniformly in (0, 1). Now Whittaker has

[4] [3, p. 454].

shown[5] that if a Lidstone series converges for some non-integral value of z, it converges for all z, uniformly in any bounded region, and so represents an entire function, which in our case must be $f(z)$. This completes the proof of Theorem 3.

By applying, instead of Lemma 1, the formula obtained by integrating

$$\int_0^1 f(x) \cos \tfrac{1}{2}k\pi x \, dx$$

by parts, we can prove Schoenberg's theorem [2], analogous to Theorem 2, that a function $f(z)$ of exponential type is a cosine polynomial if $f^{(2n)}(1) = f^{(2n+1)}(0) = 0$ $(n = 0, 1, 2, \cdots)$; and we can obtain Whittaker's result [3] corresponding to Theorem 3, concerning the expansion of $f(z)$ in a series with coefficients $f^{(2n)}(1), f^{(2n+1)}(0)$. The analogue of Theorem 1 is

THEOREM 4. *If $f(x)$ is of class C^∞ in $0 \leqq x \leqq 1$, and*

$$f^{(4n)}(x) \geqq 0, \quad (-1)^n f^{(2n)}(1) \geqq 0, \quad (-1)^n f^{(2n+1)}(0) \leqq 0, \qquad n = 0, 1, 2, \cdots,$$

then $f(x)$ coincides over $(0, 1)$ with an entire function of exponential type at most $\pi/2$.

REFERENCES

1. H. Poritsky, *On certain polynomial and other approximations to analytic functions*, Transactions of this Society, vol. 34 (1932), pp. 274–331; p. 287.

2. I. J. Schoenberg, *On certain two-point expansions of integral functions of exponential type*, this Bulletin, vol. 42 (1936), pp. 284–288.

3. J. M. Whittaker, *On Lidstone's series and two-point expansions of analytic functions*, Proceedings of the London Mathematical Society, (2), vol. 36 (1933–1934), pp. 451–469.

4. D. V. Widder, *Functions whose even derivatives have a prescribed sign*, Proceedings of the National Academy of Sciences, vol. 26 (1940), pp. 657–659.

DUKE UNIVERSITY

[5] [3, p. 455].

REPRESENTATIONS OF BOOLEAN ALGEBRAS

ORRIN FRINK, JR.

There are several proofs in the literature[1] of M. H. Stone's theorem on the representation of Boolean algebras by sets [2, 4, 5, 7, 8, 9]. This note contains a simplified version of Stone's original proof, adapted to the following set, I–IV, of postulates for a Boolean algebra B in terms of the special element 0 and the undefined operations *product* ab and *negation* b'. It is assumed that 0 is in B, and that if a, b, and c are in B, then ab and b' are in B, and

I. $ab=ba$.

II. $a(bc)=(ab)c$.

III. $aa=a$.

IV. $ab=a$ if and only if $ab'=0$.

Replacing b by a in IV gives V: $aa'=0$. Since $a0=a(aa')=(aa)a'=aa'=0$, we have VI: $a0=0$.

DEFINITIONS. *A point is a set P of elements of B such that*

α. The element 0 is not in P.

β. If a is in P and b is in P, then ab is in P.

γ. P is maximal with respect to properties α and β.

The set R_a of all points P which contain a is defined to be the representative set corresponding to the element a.

LEMMA 1. *If ab is in P, then a is in P.*

PROOF. If a were not in P, then P would not be maximal, since a and all products pa, where p is in P, could then be added to P without disturbing α, since if $pa=0$, then $pab=0$.

LEMMA 2. *If a is not equal to 0, then a is in some point P.*

PROOF. All sets of elements of B which contain a and satisfy α and β form a system S partially ordered by set inclusion. Any linearly ordered subsystem L of S has an upper bound in S, namely the union of all members of L. Hence by Zorn's lemma [10, 11], there exists in S at least one maximal element P.

THEOREM. *The correspondence between elements a of B and their representative sets R_a is an isomorphism; that is,* 1. $R_{ab}=R_a \cap R_b$; 2. $R_{a'}=C(R_a)$; 3. *if* $R_a=R_b$, *then* $a=b$.

---

[1] See also N. H. McCoy and D. Montgomery, Duke Mathematical Journal, vol. 3 (1937), pp. 455–459.

PROOF. 1. If P is in R_a and R_b, then it is in R_{ab} by β. Conversely, if P is in R_{ab}, it is in R_a and R_b by Lemma 1.

2. R_a and $R_{a'}$ are complementary, for if a point P is not in $R_{a'}$, there is an element b in P such that $a'b=0$, since otherwise a' and products $a'p$ could be added to P, and P would not be maximal. Hence by IV, $ab=b$; therefore ab is in P. Then by Lemma 1, a is in P, and P is in R_a. On the other hand, R_a and $R_{a'}$ have no common point P, since such a P would have to contain $aa'=0$.

3. If $a\neq b$, then either $ab'\neq 0$ or $a'b\neq 0$, since otherwise by IV $a=ab=b$. If $ab'\neq 0$, then by Lemma 2, ab' is in some point P. By Lemma 1, a is in P. But b is not in P, since $ab'b=0$. Likewise if $a'b\neq 0$, there is a point containing b but not a. Hence $R_a\neq R_b$.

COROLLARY. *The set* I–IV *is an adequate postulate system for Boolean algebras.*

For I–IV hold in any Boolean algebra, and any algebra in which they hold has been shown to be isomorphic to an algebra of sets, and hence to a Boolean algebra. This postulate system is comparable in simplicity with other well known sets [1, 3, 6].

REFERENCES

1. B. A. Bernstein, *A set of four independent postulates for Boolean algebras*, Transactions of this Society, vol. 17 (1916), pp. 50–52.

2. Garrett Birkhoff, *Lattice Theory*, American Mathematical Society Colloquium Publications, vol. 25, New York, 1940, p. 89.

3. E. V. Huntington, *New sets of independent postulates for the algebra of logic*, Transactions of this Society, vol. 35 (1935), pp. 274–304, 557–558, 971.

4. S. Kakutani, *Weak topology, bicompact sets, and the principle of duality*, Proceedings of the Imperial Academy of Japan, vol. 16 (1940), pp. 63–67.

5. E. Livenson, *On the realization of Boolean algebras by algebras of sets*, Recueil Mathématique (Mathematicheskii Sbornik), new series, vol. 7 (1940), pp. 309–312.

6. M. H. Stone, *Postulates for Boolean algebras and generalized Boolean algebras*, American Journal of Mathematics, vol. 57 (1935), pp. 703–732.

7. ———, *The theory of representations for Boolean algebras*, Transactions of this Society, vol. 40 (1936), pp. 37–111.

8. ———, *The representation of Boolean algebras*, this Bulletin, vol. 44 (1938), pp. 807–816.

9. H. Teresaka, *Über die Darstellungen der Verbände*, Proceedings of the Imperial Academy of Japan, vol. 14 (1938), pp. 306–311.

10. J. W. Tukey, *Convergence and Uniformity in Topology*, Princeton, 1940, p. 7.

11. M. Zorn, *A remark on method in transfinite algebra*, this Bulletin, vol. 41 (1935), pp. 667–670.

THE PENNSYLVANIA STATE COLLEGE

A FIXED-POINT THEOREM FOR TREES[1]

A. D. WALLACE

By a *tree* we mean a compact (=bicompact) Hausdorff space which is acyclic in the sense that

(i) *if* $\mathfrak{U}$ *is a f.o.c.* (=*finite open covering*) *of a tree* T *then there is a f.o.c.* $\mathfrak{V} \subset \mathfrak{U}$ *such that the nerve* $N(\mathfrak{V})$ *is a combinatorial tree,*

and which is locally connected in the sense that

(ii) *if* $\mathfrak{U}$ *is a f.o.c. of* T *then there is a f.o.c.* $\mathfrak{V} \subset \mathfrak{U}$ *whose vertices are connected sets.*

It may be shown [3] that an acyclic continuous curve in the usual sense is a tree in our terminology. If q is a mapping which assigns to each point t of a topological space a *set* qt in a topological space, then we say that q is *continuous* provided that for each t and each neighborhood U of qt we can find an open set V containing t such that if t' is in V then qt' is in U. Our present purpose is to establish the following result:

(A) *Let* T *be a tree and let* q *be a continuous point-to-set mapping which assigns to each point* t *a continuum* qt *in* T. *Then there is a* $t_0 \in T$ *such that* $t_0 \in qt_0$.

The proof (which is divided into several lemmas) uses strongly a technique introduced by H. Hopf [1]. However the present note has been made self-contained.

(A_1) *The intersection of two continua of* T *is again a continuum.*

PROOF. Let B_1, B_2 be two continua such that $B_1 \cdot B_2 = C_1 + C_2$ where the C_i are disjoint and closed. We can find disjoint open sets $D_i \supset C_i$. Let $t \in T - B_1 \cdot B_2$. We can then find an open set V_t containing t and which does not meet both B_1 and B_2. The sets D_i together with the sets V_t can be reduced to a f.o.c. $\mathfrak{U}$ of T. Let $\mathfrak{V} \subset \mathfrak{U}$ be the f.o.c. described in (i). Let $\mathfrak{V}_i$ be those vertices of $\mathfrak{V}$ on B_i. It is easy to see that $N(\mathfrak{V}_i)$ is connected. If $c_i \in C_j$ we can find a chain of 1-cells E_i in $N(\mathfrak{V}_i)$ whose first vertex contains c_1 and whose last vertex contains c_2. Now we cannot have $E_i \subset D_1 + D_2$ and E_i contains a vertex which is not on B_j. Hence $E_1 \neq E_2$ and so $N(\mathfrak{V})$ is not a tree. This contradiction completes the proof.

[1] Presented to the Society, May 3, 1941.

(A_2) *Any f.o.c.* $\mathfrak{U}$ *of* T *contains a f.c.c.* $\mathfrak{F}\subset\mathfrak{U}$ *so that each* $F_i\in\mathfrak{F}$ *is connected and further* $N(\mathfrak{F})$ *is a combinatorial tree.*

PROOF. We can find a f.o.c. $\mathfrak{V}\subset\mathfrak{U}$ such that $N(\mathfrak{V})$ is a tree. By a lemma due to Čech [5, p. 180] we can find a f.c.c. $\mathfrak{F}'\subset\mathfrak{V}$ such that $\mathfrak{F}'$ and $\mathfrak{V}$ are combinatorially isomorphic. Let $\mathfrak{R}_i$ be the f.o.c. $(V_i, T-F_i')$. Using (ii) it is easy to see that there is a f.o.c. $\mathfrak{W}$ such that each W_i is connected and $\overline{\mathfrak{W}}\subset\mathfrak{R}_i$, for each i. Let i be fixed. If W_j meets F_i' then so does $\overline{W}_j$ and so is contained in V_i. Let Q_i be the union of all such W_j. Then the closure of this set has a component-wise decomposition, say $\overline{Q}_i=F_{i1}+F_{i2}+\cdots+F_{is_i}$. Let $\mathfrak{F}$ be the f.c.c. $\{F_{ij}\}$. It is clear that the elements of $\mathfrak{F}$ are connected and it is not hard to show that $\dim\mathfrak{F}\leqq 1$, that is, at most two elements of $\mathfrak{F}$ have a non-null intersection. If we have a chain

$$F_{i_1j_1}, F_{i_2j_2}, \cdots, F_{i_rj_r}, F_{i_1j_1}, \qquad r>2,$$

such that each set meets the following but such that there are no other intersections, then the sets $F_{i_1j_1}$ and $\sum_{s>1}F_{i_sj_s}$ are connected and therefore by (A_1) so is their meet, the set $F_{i_1j_1}\cdot F_{i_2j_2}+F_{i_1j_1}\cdot F_{i_rj_r}$. But then we would have $F_{i_1j_1}\cdot F_{i_2j_2}\cdot F_{i_rj_r}\neq 0$, a contradiction. It follows that $N(\mathfrak{F})$ is a tree.

(B) *Let* q *be a mapping which assigns to each continuum* K *in* T *a continuum* qK *in* T *such that if* $K_1\subset K_2$, *then* $qK_1\subset qK_2$. *If* $\mathfrak{F}=\{F_i\}$ *is a f.c.c. with connected sets such that* $N(\mathfrak{F})$ *is a tree then there is an* F_i *for which* $F_i\cdot qF_i\neq 0$.

PROOF. Let $N=N(\mathfrak{F})$ and suppose that the vertices of N are e_i. To each i we assign an i' so that $F_{i'}$ meets qF_i. We then have a mapping $e_i\to e_{i'}$ and since N is a tree it follows at once by a result due to Hopf [1, Lemma γ] that we can find an edge e_me_n which is contained in the chain joining $e_{m'}$ to $e_{n'}$.[2] We show that $F_k\cdot qF_k\neq 0$, $k=m$, or n. We have $F_m\cdot F_n\neq 0$ and by construction $F_{m'}\cdot qF_m\neq 0\neq F_{n'}\cdot qF_n$. Further

$$(*) \qquad F_{m'}, F_i, \cdots, F_m, F_n, F_j, \cdots, F_{n'}$$

is a simple chain of sets. Of course it may happen that F_n precedes F_m in (*) but this is of no importance. Let X be the union of all the sets in (*) from $F_{m'}$ up to and including F_m. Let Y be similarly defined for the other part of (*). Then X and Y are continua with $X\cdot Y=F_m\cdot F_n$.

[2] I am indebted to Professor S. Lefschetz for the remark that $e_i\to e_{i'}$ generates a chain-mapping (that is, a mapping permutable with the boundary operator) if we define for the image of e_me_n the chain joining e_m to e_n. Since N is acyclic it follows at once that there is a fixed element. This may replace the result of Hopf.

Also F_m+F_n is a continuum and so is $Z=qF_m+qF_n$. Clearly Z meets the end-vertices of (*). By (A_1) $Z\cdot(X+Y)$ is a continuum. Hence $Z\cdot X\cdot Y$ is not null. Thus $F_m\cdot F_n\cdot(qF_m+qF_n)\neq 0$ and this completes the proof of (B).

It is not hard to see that if q is a mapping of the type described in (A) then q satisfies the conditions in (B) if we define $qK=\sum qt$, $t\in K$, for each continuum K of T. The proof is quite similar to those for analogous results concerning single-valued mappings.

We now turn to a proof of (A). Suppose that no t is in qt. We can find a neighborhood R_t of t so that $\overline{R}_t$ does not meet qt. Let $V_t=T-\overline{R}_t$. Since $qt\subset V_t$ we can find a neighborhood S_t of t so that $t'\in S_t$ implies $qt'\subset V_t$. Let U_t be the meet of R_t and S_t. We cover T by a finite subcollection $\{U_i\}=\{U_{t_i}\}$ of the sets U_t. We can find a refinement $\mathfrak{F}$ of $\mathfrak{U}=\{U_i\}$ which satisfies the conditions in (B) in consequence of (A_2). By (B) we can find a set F in $\mathfrak{F}$ so that F meets qF. In other words we find a t in F such that F meets qt. Now F is in some U_i and hence qt is in the corresponding V_i. But since F does not meet the set V_i it cannot meet qt. This contradiction completes the proof.

A continuous transformation $fM=N$ is said to be *free* (Hopf [1]) provided there is a continuous transformation $gM\subset M$ such that $fgx\neq fx$ for each $x\in M$. The transformation f is *monotone* if the set $f^{-1}y$ is connected for each $y\in N$.

(C) *No continuum admits a free monotone transformation onto a tree.*

PROOF. Let $fM=T$ be monotone and $gM\subset M$ be continuous. For each $t\in T$ we set $qt=fgf^{-1}t$. It is not hard to see that q is continuous and hence we may apply (A). But from $t\in qt$ it follows at once that there is an $x\in M$ with $fgx=fx$.

The transformations $fM\subset N$ and $gM\subset N$ have a *coincidence* (Lefschetz [2]) if there is an $x\in M$ with $fx=gx$. As in (C) we may show that

(D) *A monotone transformation $fM=T$ of a continuum onto a tree admits a coincidence with any continuous transformation $gM\subset T$.*

Remarks. The result (A) is usually called the Scherrer fixed-point theorem when q is single-valued and T is an acyclic continuous curve. For a list of papers concerning it see Hopf [1]. Corollary (C) will be found in [3]. The result (A) was found while constructing a proof of (D). Finally (A) is analogous to a result of S. Kakutani [4] who has shown that if S is an n-simplex and to each $s\in S$ we assign continuously a closed convex set qs then there is an $s_0\in qs_0$.

BIBLIOGRAPHY

1. H. Hopf, Fundamenta Mathematicae, vol. 28 (1937), p. 31.
2. S. Lefschetz, *Topology*, American Mathematical Society Colloquium Publications, vol. 12, New York, 1930.
3. A. D. Wallace, Duke Mathematical Journal, vol. 6 (1940), p. 31.
4. S. Kakutani, *A generalization of Brouwer's fixed-point theorem*, to appear in the Duke Mathematical Journal.
5. E. Čech, Fundamenta Mathematicae, vol. 25 (1932), p. 149.

PRINCETON UNIVERSITY

ON THE DEFINITION OF CONTACT TRANSFORMATIONS

ALEXANDER OSTROWSKI

If z is a function of $x_1, \cdots, x_n$ and $p_\nu = \partial z/\partial x_\nu$, $\nu = 1, \cdots, n$, a *contact transformation* in the space of $z, x_1, \cdots, x_n$, is defined by a set of $n+1$ equations

$$\text{(a)} \qquad Z = Z(z, x_\mu, p_\mu), \qquad X_\nu = X_\nu(z, x_\mu, p_\mu), \qquad \nu = 1, \cdots, n,$$

such that *firstly* in calculating the n derivatives

$$P_\nu = \frac{\partial Z}{\partial X_\nu}, \qquad \nu = 1, \cdots, n,$$

the expressions for the P_ν are given by a set of n equations

$$\text{(b)} \qquad P_\nu = P_\nu(z, x_\mu, p_\mu), \qquad \nu = 1, \cdots, n,$$

in which the derivatives of the p_μ *fall out*; and *secondly* the equations (a) and (b) can be resolved with respect to z, x_μ, p_μ:

$$\text{(A)} \qquad z = z(Z, X_\mu, P_\mu), \qquad x_\nu = x_\nu(Z, X_\mu, P_\mu), \qquad \nu = 1, \cdots, n,$$

$$\text{(B)} \qquad p_\nu = p_\nu(Z, X_\mu, P_\mu), \qquad \nu = 1, \cdots, n.$$

These two postulates are equivalent with the hypothesis that the $2n+1$ equations (a), (b) form a transformation between the two spaces of the sets of $2n+1$ independent variables (z, x_ν, p_ν), (Z, X_ν, P_ν) satisfying the Pfaffian condition

$$dZ - \sum_{\nu=1}^{n} P_\nu dX_\nu = \rho\left(dz - \sum_{\nu=1}^{n} p_\nu dx_\nu\right), \qquad \rho \neq 0.$$

In the following lines we prove: *the hypothesis that the system* (A) *is a corollary of the system* (a) *and conversely is already sufficient in order that* (a) *define a contact transformation*, that is to say: under this hypothesis the expressions (b) of P_ν, derived from (a), are independent of the second derivatives of z.

As to the functions $Z(z, x_\mu, p_\mu)$, $X_\nu(z, x_\mu, p_\mu)$, $z(Z, X_\mu, P_\mu)$, $x_\nu(Z, X_\mu, P_\mu)$, we shall assume:

(1) that the functions $Z(z, x_\mu, p_\mu)$, $X_\nu(z, x_\mu, p_\mu)$ possess continuous partial derivatives of the first order with respect to their $2n+1$ arguments;

(2) that the "total Jacobian"

$$\left|\frac{dX_\nu}{dx_\mu}\right|, \qquad \nu, \mu = 1, \cdots, n, \tag{1}$$

does not vanish identically in the $(2n+1)+n(n+1)/2$ variables z, x_ν, p_ν, p'_{x_μ}. Here the "*total derivative*" with respect to x_ν is defined by

$$\frac{d}{dx_\nu} = p_\nu\frac{\partial}{\partial z} + \frac{\partial}{\partial x_\nu} + \sum_{\mu=1}^{n} p'_{\mu x_\nu}\frac{\partial}{\partial p_\mu}, \qquad \nu = 1, \cdots, n; \tag{2}$$

(3) that the functions $z(Z, X_\mu, P_\mu)$, $x(Z, X_\mu, P_\mu)$ possess continuous partial derivatives of the first order with respect to their $2n+1$ arguments. (This hypothesis is certainly satisfied if the functions $Z(z, x_\mu, p_\mu)$, $X_\nu(z, x_\mu, p_\mu)$ possess continuous partial derivatives of the *second* order with respect to their arguments and if the determinant (1) does not vanish.)

From these three hypotheses it follows at once that the determinant $|dx_\nu/dX_\mu|$, $\nu, \mu = 1, \cdots, n$, does not vanish identically, since $x_1, \cdots, x_n$ can be assumed as being independent variables.

Then, if $Z(z, x_\mu, p_\mu)$, $X_\nu(z, x_\mu, p_\mu)$ were all free of the p_μ, we have obviously a reversible point-to-point transformation between the space of $n+1$ variables (z, x_ν) and that of $n+1$ variables (Z, X_ν). And the same result holds if $z(Z, X_\mu, P_\mu)$, $x_\nu(Z, X_\mu, P_\mu)$ were all free of the P_μ. We may therefore assume without loss of generality that p_μ do actually appear in the equations (a) and P_μ in the equations (A).

By means of total derivatives (2), P_μ can be calculated from the n equations

$$\frac{dZ}{dx_\nu} = \sum_{\mu=1}^{n} P_\mu\frac{dX_\mu}{dx_\nu}, \qquad \nu = 1, \cdots, n. \tag{3}$$

Consider the n expressions

$$B_\nu = \frac{\partial Z}{\partial p_\nu} - \sum_{\mu=1}^{n} P_\mu \frac{\partial X_\mu}{\partial p_\nu}, \qquad \nu = 1, \cdots, n, \tag{4}$$

and suppose first that not all B_ν vanish.

Then, if for instance $B_1 \neq 0$, let

$$q_\lambda = \frac{\partial p_\lambda}{\partial x_1} = \frac{\partial p_1}{\partial x_\lambda}, \qquad \lambda = 1, \cdots, n.$$

In differentiating (3) with respect to q_λ we have easily

$$\sum_{\mu=1}^{n} P'_{\mu q_\lambda} \frac{dX_\mu}{dx_\nu} = \delta_\nu^\lambda B_1 + \delta_\nu^1 (1 - \delta_\lambda^1) B_\lambda, \qquad \nu, \lambda = 1, \cdots, n,$$

where as usual

$$\delta_\nu^\mu = \begin{cases} 0, & \mu \neq \nu, \\ 1, & \mu = \nu. \end{cases}$$

But now it follows that

$$\frac{\partial(P_1, \cdots, P_n)}{\partial(p'_{1x_1}, \cdots, p'_{nx_1})} \left| \frac{dX_\mu}{dx_\nu} \right| = \left| \delta_\nu^\lambda B_1 + \delta_\nu^1 (1 - \delta_\lambda^1) B_\lambda \right| = B_1^n \neq 0,$$

the P_ν are independent with respect to $p'_{1x_1}, \cdots, p'_{nx_1}$, and the equations (A) are only possible, if they do not contain the P_ν at all, the case which has been already discarded.

We have therefore $B_\nu = 0$, $\nu = 1, \cdots, n$. Then the equations (3) and (4) reduce to the $2n$ equations

$$\begin{aligned} \frac{\partial Z}{\partial x_\nu} + p_\nu \frac{\partial Z}{\partial z} &= \sum_{\mu=1}^{n} P_\mu \left(\frac{\partial X_\mu}{\partial x_\nu} + p_\nu \frac{\partial X_\mu}{\partial z} \right), && \nu = 1, \cdots, n, \\ \frac{\partial Z}{\partial p_\nu} &= \sum_{\mu=1}^{n} P_\mu \frac{\partial X_\mu}{\partial p_\nu}, && \nu = 1, \cdots, n. \end{aligned} \tag{5}$$

On the other hand, the rank of the matrix with n columns and $2n$ rows

$$\begin{pmatrix} \dfrac{\partial X_\mu}{\partial x_\nu} + p_\nu \dfrac{\partial X_\mu}{\partial z} \\ \dfrac{\partial X_\mu}{\partial p_\nu} \end{pmatrix}, \qquad \mu, \nu = 1, \cdots, n,$$

is n, since otherwise (1) would vanish. We see that in this case P_ν *can be expressed from* (5) *by* z, x_μ, p_μ.

Since the same argument applies to the equations (A), p_ν can be expressed by means of Z, X_μ, P_μ.

We have now the 4 sets of relations (a), (b), (A), (B). It is easily seen that the $2n+1$ relations (A), (B) are inverse of the $2n+1$ relations (a), (b), if p_μ resp. P_μ are considered as independent variables. Indeed, in putting the values (a) and (b) in the relations (A), (B), we must obtain identities $z=z$, $x_\nu=x_\nu$, $p_\nu=p_\nu$, for otherwise a non-identical relation between z, x_μ, p_μ would follow, that is, a differential equation, satisfied by an "arbitrary" function $z(x_1, \cdots, x_n)$.

We see that in the case of *one* function of n variables a reversible transformation of the first order is necessarily a contact transformation.

Our implicit definition of the "reversible transformations of the first order" leads to non-trivial results in the cases in which the contact transformations in the usual sense do not exist at all. For instance, in the case of $n>1$ functions $z_1(x), \cdots, z_n(x)$ of one independent variable, all contact transformations reduce simply to the point-to-point transformations in the space of $n+1$ variables $z_1, \cdots, z_n, x$. On the other hand, there exist in this case non-trivial reversible transformations. If for instance

$$X = z_n - x\sum_{\nu=1}^{n} p_\nu, \qquad Z_\lambda = z_\lambda, \quad (\lambda = 1, \cdots, n-1), \qquad Z_n = -\sum_{\nu=1}^{n} p_\nu,$$

$$p_\nu = \frac{dz_\nu}{dx}, \qquad P_\nu = \frac{dZ_\nu}{dX}, \qquad \nu = 1, \cdots, n,$$

we have easily for $\lambda=1, \cdots, n-1$

$$\frac{P_\lambda}{xP_n - 1} = \frac{p_\lambda}{\sum_{\kappa=1}^{n-1} p_\kappa}, \qquad x = \frac{1 + \sum_{\lambda=1}^{n-1} P_\lambda}{P_n},$$

and therefore

$$z_n = X - \frac{Z_n}{P_n}\left(1 + \sum_{\lambda=1}^{n-1} P_\lambda\right), \qquad z_\lambda = Z_\lambda, \qquad \lambda = 1, \cdots, n-1.$$

We have determined in the case of n functions of one variable all reversible transformations of the first order by means of certain Pfaffian and Mongeian relations. These results will be exposed in another paper.

UNIVERSITY OF BASEL

FOURTEEN SPECIES OF SKEW HEXAGONS

H. S. WHITE

1. **Hexagon and hexahedron.** For a tentative definition, let a skew hexagon be a succession of six line segments or edges, finite or infinite, the terminal point of each being also the initial point of another, the terminus of the last being the first initial point. Secondly, let there let be six marks to designate the points, the six to be in a fixed order. Further, let the first point be any one of the six, so that only the cyclic order is essential. At each of the six points, pass a plane through the two lines, and denote the plane by the same mark as that point. Adjoin to the figure these six entire planes, a complete hexahedron.

2. **Sets of 64 hexagons.** The initial and terminal points of an edge denote ambiguously two segments, one finite and one infinite. If the marks in their order are 123456, then that cycle may denote equally any one of 64 different hexagons, $2^6 = 64$, obtained by selection. If desired, the one whose edges are all finite may be taken to represent the set. But we can define positive and negative directions along each line, arbitrarily on one, then by inference on all other lines of intersection of the planes. Thereafter it will be natural to understand by 12 a positive segment on the line, by 21 a negative.

3. **Hexagons and cyclic permutations.** As above defined, every skew hexagon determines one and only one set of six planes, but these intersect, not in 6 but in 15 lines, and so contain many different hexagons. Any cyclic permutation of the marks of the planes, as 154263, may be interpreted as describing a hexagon. An edge is given by any four consecutive marks; for example, 2631 will denote a segment on the intersection-line 63, from its point in plane 2 to its point in plane 1. Thus a cycle reversed will denote the same edges in reverse order and we shall consider the two as identical. As there are 120 cyclic permutations of 123456, this gives us only 60 different hexagons to assort into species.

All sets of 6 planes will be defined as equivalent, provided no four have a common point. By moving planes freely any one can be made to coincide with any other set, without passing through the excluded positions. It is well known that the traces of five planes in a sixth bound in that sixth one convex pentagon as well as certain triangles and quadrilaterals. Of the six pentagons, each has two edges which are edges of other pentagons, thus providing a natural order of the planes. Moreover those six common edges form a continuous

broken line or skew hexagon, which shall be called the *normal hexagon* of that set of planes. Let its vertices and planes be marked, as described above, in their order along the normal hexagon, 1, 2, 3, 4, 5, 6, the vertex 1 being any one taken at random. This gives 6 equivalent notations for the set; in other words, *the set of* 6 *planes is equivalent to itself under the cyclic permutations of the normal hexagon.* Consequently, any other skew hexagon occurring in the figure is equivalent to that derived from it by cyclic permutation: $S=(123456)$, of the marks of its 6 planes and vertices. Note that the planes take their marks from the vertices of the normal hexagon, after which the vertices of any other hexagon in the figure (having two adjacent edges in each plane) take their marks each from the plane that contains the two edges at that point.

For example, 146235 will denote a skew hexagon different from the normal. The operations repeated give the 5 equivalent cycles:

2	5	1	3	4	6,
3	6	2	4	5	1,
4	1	3	5	6	2,
5	2	4	6	1	3,
6	3	5	1	2	4.

Thus there are found sets of 6 equivalent hexagons, but also some of fewer than 6. For example 153426 is in a set of only 3.

4. **Survey of all species of skew hexagons.** On any single set of six planes will be found all possible species of hexagons, for any given

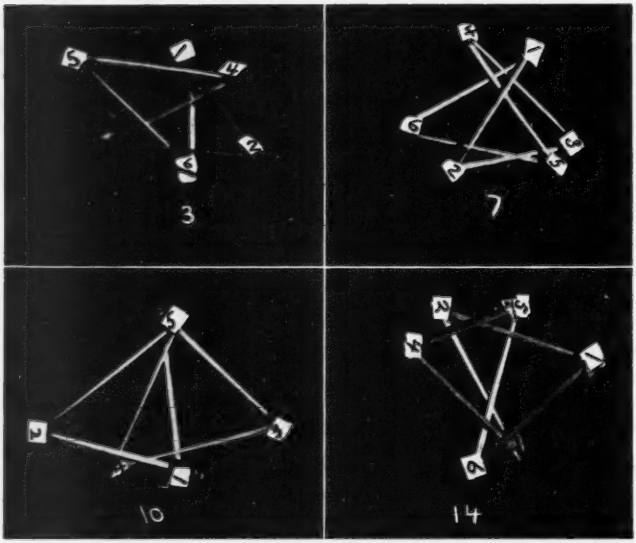
TABLE I

No.	In Set	Cycle	No.	In Set	Cycle
1	1	123456	8	6	132645
2	6	146235	9	3	123654
3	6	132654	10	3	135264
4	6	125364	11	3	135462
5	6	142356	12	3	132564
6	6	143562	13	3	142365
7	6	162345	14	2	143652

hexagon, nonsingular, gives rise to a set of 6 planes, and all such sets are equivalent. If the planes have been referred to their normal hexagon, then all 60 hexagons that lie in all six planes are represented in Table 1 above, a single cycle representing six, or three, or two types

of 64 that share a single set of 6 lines, or the one unique normal type.

5. **A positive order on each line.** Suppose the vertices of the normal hexagon to be marked in the order of their occurrence along its periphery: 123456. Each segment, as 12, is known to be tangent at some



EXPERIMENTAL MODELS, CRUDE BUT SERVICEABLE, OF SKEW HEXAGONS

No. 3, Normalized cycle? 143265.
No. 7, Normalized cycle? 164532.
No. 10, Normalized cycle? 146253.
No. 14, Normalized cycle? 125634.

one of its points to a conic enveloping, in plane 1, the traces of the other planes in the order, which shall be chosen as *positive* order, of their arrangement about the unique convex pentagon formed by those traces, say 23456, and indeed it touches on the same segment the line conics in planes 1 and 2. Then on any other line, as 5, in plane 1 the intersections of the others must occur in the same order, 2346, omitting its own mark 5. So on all the lines in plane 1 we have a

"normal" assignment of marks. Then since each line lies in two planes, the marking is automatically extended to all the lines in all 6 planes.

6. **A character of every hexagon.** On the lines of the normal hexagon the external segment must contain two intersections of other

TABLE II

Species	Character	Species	Character
1	000000	8	120102
2	002220	9	221001
3	221102	10	001221
4	001210	11	110110
5	000112	12	211101
6	010201	13	101112
7	001010	14	020202

planes. And since the internal segment is designated by terminals in their positive order, the external will be negative from the same initial point to the same terminus. Give it therefore two marks—and 2. Extending this scheme, assign to each directed segment a sign + or −, and either 0, 1, or 2 to indicate how many planes cut it internally. Every hexagon will then have six double "characters" in a definite cyclic order, the normal hexagon being described by $(0^+ 0^+ 0^+ 0^+ 0^+ 0^+)$. We have now to list the characters of all 14 species for comparison.

A segment of such a hexagon being given by an ordered tetrad, as 4265, we have yet to choose between the positive and the negative direction from 4 to 5. *Always we shall take the positives for a typical representative of the* 64 *which share the same designation* (cf. §4). With this convention, we need write only the character 0, 1, or 2 for each segment of the hexagon. In the above example, on the line 26 the marks 2 and 6 are not in intersecting planes, and the two intersections 1, 3 do not fall on the positive segment from 4 to 5. The mark will be 0. Again, for a tetrad 2615 it is 2, for 6413 it is 1, and so on. For the 14 species the characters of the representative hexads are shown in Table II.

These characters are not quite distinctive, but nearly so. Sets 3 and 5 differ as right-hand and left-; also sets 12 and 13. But not so sets 8, 9, 10, and 14: each is auto-symmetric, and a change of sense will not exchange them. Distinguish odd and even triads of planes, and we see that triads are collected in No. 10, alternated in sequence in No. 9.

There are, therefore, 14 *types among these* 60 *skew hexagons, all lying*

in the same set of 6 planes, two segments in each plane. No two types are similar as regards distribution of intersection-points; but two pairs are symmetrical to each other, and four sets are auto-symmetric.

THEOREM. *The character of every skew hexagon must contain no aces, or two, or four.* (A synthetic proof seems desirable.)

7. **Determining the species of a concrete model.** When a given material model is presented for classification, one first assigns marks, as 123456 to the vertices in their order along the perimeter, first point and direction being chosen at random. Next, the cyclic order of points on each side is found by direct observation. From this, the determination of a *normal order* is most readily shown by an example.

From a certain specimen we take these sequences:

On line	*Order*	*On line*	*Order*
12	6453 or 6354	45	3162 or 3261
23	1654 or 1456	56	4321 or 1234
34	2615 or 2516	61	5423 or 5324

To unite these consistently in a hexad, take the first and third. They require 264153 or 261453. Comparison with the order on 23 decides in favor of 261453. Adopt this for *normal order*. Apply the substitution:

$$T = \begin{pmatrix} 261453 \\ 123456 \end{pmatrix},$$

by which the normal order becomes 123456. This redesignates the points, and the given hexagon is now entitled $T(123456) = 316452$. This has the *character* 202121. As this is not found in Table II, adopt the reverse normal order, and find for character 020101. This is No. 6 on the list. Verify by applying the operator $S = (123456)$ five times to 316452. The last member of the set is 265341, readily recognizable as the reverse of No. 6 which was 143562. As errors are easy in calculating the character, such a final check should always be demanded. Another example, if desired for practice, is taken from a second model. The orders observed are: 6543, 5146, 2156, 3612, 4132, 5324, leading to No. 8 in the table.

This plan of classification is hardly practicable for skew n-gons when $n > 6$, owing to the rapid increase in number of species. For a single species of n-gon, whose vertices are on a twisted cubic curve, it could be generalized.

VASSAR COLLEGE

ON THE EXISTENCE OF ELECTRICAL NETWORKS

C. E. CLARK

This paper is concerned with the existence of electrical networks which satisfy certain preassigned conditions. These conditions have to do with the existence of circuits with preassigned resistances in common.

Consider a finite set of points in euclidean 3-space and a set of straight line segments joining pairs of these points. Furthermore, suppose that no two of the segments intersect at an interior point, but any number of segments may have a common end-point. Each of the segments is called a *branch* of the graph. With each branch of the graph let there be associated a non-negative real number called the *resistance* of the branch. The graph together with the resistances is an *electrical network*. A *circuit* of a network is a topological circle of the network together with an orientation of this circle.

Let two circuits C_i and C_j have the branches b_{ij}^p, $p=1, 2, \cdots$, in common. Let r_{ij}^p be the resistance of b_{ij}^p. Let $\bar{r}_{ij}^p = r_{ij}^p$ if the orientations of C_i and C_j agree along b_{ij}^p while $\bar{r}_{ij}^p = -r_{ij}^p$ if the orientations are opposite. Then $I_{ij} = \sum_p \bar{r}_{ij}^p$, $p=1, 2, \cdots$, is the *intersection* of C_i and C_j. We see that I_{ii} is the sum of the resistances of the branches of C_i.

Let C_i, $i=1, 2, \cdots, n$, be distinct circuits of a network. Then the matrix $\|I_{ij}\|$, $i, j=1, \cdots, n$, is the *intersection matrix of the* C_i. This matrix is symmetric and has non-negative diagonal elements. Any matrix with these two properties is an *intersection matrix*. An intersection matrix M is *realizable* when there exists a network which has a set of circuits whose intersection matrix is M.

THEOREM 1. *Given an intersection matrix*

$$\|I_{ij}\|, \qquad i, j = 1, \cdots, n; \tag{1}$$

if $I_{ii} \geqq \sum_j |I_{ij}|$, $j \neq i$, $i=1, \cdots, n$, *then* (1) *is realizable.*

PROOF. Choose $2n$ points on each of n oriented circles S_i, $i=1, \cdots, n$. These points divide each S_i into $2n$ branches. Denote the $2n$ branches of S_i by $b_{i1}, o_{i1}, b_{i2}, o_{i2}, \cdots, b_{in}, o_{in}$ with the b's separated by the o's. To each o is assigned the resistance zero. We shall next identify b_{ij} and b_{ji}, $i \neq j$.[1] In the resulting figure there will be a circle that is the natural topological image of S_i. We shall say that this circle *is* S_i.

[1] I.e., we bring the two end-points of b_{ij} into coincidence with the end-points of b_{ji} and replace these two branches by a single branch. To do this it may be necessary to replace any figure by its homeomorph.

The identification of b_{ij} and b_{ji} can be performed in two ways. If $I_{ij}>0$, the identification is made so that S_i and S_j agree in orientation along their common branch; if $I_{ij}<0$, S_i and S_j have opposite orientations along their common branch; and if $I_{ij}=0$, we choose either identification. To the common branch is assigned the resistance $|I_{ij}|$.

To b_{ii}, $i=1, \cdots, n$, is assigned the resistance $I_{ii}-\sum_j |I_{ij}|$, $j\neq i$.

In the resulting network the S_i, $i=1, \cdots, n$, are circuits whose intersection matrix is (1).

THEOREM 2. *If matrix* (1) *is realizable, and if a matrix* (1a) *is obtained from* (1) *by increasing a diagonal term* I_{ii} *to* I_{ii}', *then* (1a) *is realizable.*

PROOF. Consider a realization of (1) and let C_i be the circuit whose total resistance is I_{ii}. We break C_i at one of its vertices A and insert two branches b_i and b between the two ends A_1 and A_2 of the broken circuit. Any other branch that had been connected at A may now be connected at either A_1 or A_2. To the branch b_i is assigned the resistance $I_{ii}'-I_{ii}$, while to b is assigned the resistance zero. The new network gives a realization of (1a).

The above theorems show that if all except the diagonal elements of a symmetric matrix are given, then it is possible to assign the diagonal elements so that the resulting matrix is realizable. Also there are minimal sets of diagonal elements that can be assigned. But there is not a unique minimal set of diagonal elements as we now show by example.

We wish to find the smallest a, b, and c so that

$$\left\| \begin{matrix} a & 1 & 1 \\ 1 & b & -1 \\ 1 & -1 & c \end{matrix} \right\| \tag{2}$$

is realizable. We can take $a=1$. To do so we must have the second and third circuits S_2 and S_3 intersecting the first circuit S_1 along the same branch and both S_2 and S_3 agreeing in orientation with S_1 along this branch. Since $I_{23}=-1$, we must have S_2 and S_3 intersecting along another branch. This branch can have resistance 2 with S_2 and S_3 oppositely oriented here. This gives a realization of (2) with $a=1$, $b=3$, and $c=3$.

Also we can see that if $a<2$, then one other of the diagonal terms must be greater than 2. Hence if $a=b=c=2$ makes (2) realizable, then this set of diagonal elements is a minimal set (in the sense that one element can not be decreased without increasing another). But

by Theorem 1 the values $a=b=c=2$ make (2) realizable. Hence there is not a unique minimal set of values for a, b, and c.

THEOREM 3. *Given I_{ij}, $i\neq j$, of* (1) (*but not the diagonal elements*), *let a be fixed; then we can take $I_{aa}=\max |I_{aj}|$, $j\neq a$, and find values for I_{ii}, $i\neq a$, so that* (1) *is realizable.*

The proof is similar to the proofs of the preceding theorems.

PURDUE UNIVERSITY

ON THE SPHERICAL SURFACE OF SMALLEST RADIUS ENCLOSING A BOUNDED SUBSET OF n-DIMENSIONAL EUCLIDEAN SPACE[1]

L. M. BLUMENTHAL AND G. E. WAHLIN

1. **Introduction.** An $(n-1)$-dimensional spherical surface $S_{n-1,r}$ is the "surface" of an n-dimensional sphere of radius r in E_n, the n-dimensional euclidean space. A given spherical surface encloses M, a subset of E_n, provided M is contained in the sphere with this surface, while M is enclosable by a given $S_{n-1,r}$ whenever M is a subset of a sphere whose surface is congruent with $S_{n-1,r}$. The purpose of this article is to show (1) *if M is any bounded subset of E_n* (*containing more than a single point*) *there exists a unique $S_{n-1,r}$ of smallest radius r enclosing M* and (2) *if d is the diameter of M, then the radius of the unique smallest $S_{n-1,r}$ enclosing M satisfies the relation $r\leqq[n/2(n+1)]^{1/2}\cdot d$.*

In a proof that abounds with algebraic difficulties, H. W. E. Jung established these results in his dissertation (1901) for the case of finite point sets and indicated their extension to infinite sets at the end of his long paper.[2] Returning to the subject eight years later, Jung attempted a geometric proof for the case of *n points in a plane*, but succeeded in obtaining in this later article only necessary conditions on the smallest circle enclosing a plane (finite) set, since his procedure yields the smallest circle only in case one is assured of the existence of such a circle.[3] Though this fact can readily be supplied, the geometric considerations used by Jung are not easily extended to finite

[1] Presented to the Society, February 22, 1941.

[2] H. W. E. Jung, *Ueber die kleinste Kugel, die eine räumliche Figur einschliesst*, Journal für die reine und angewandte Mathematik, vol. 123 (1901), pp. 241–257.

[3] H. W. E. Jung, *Ueber den kleinsten Kreis, der eine ebene Figur einschliesst*, ibid., vol. 137 (1909), pp. 310–313.

point sets of n-dimensional space, while for infinite sets (even of the plane) some of the argument is not valid.[4]

It seems worthwhile, therefore, to present a simple proof of the interesting assertions (1) and (2) which avoids the analytical complexities featuring Jung's demonstration and more recent proofs (see §4), deals directly with the general case of subsets of E_n in an elementary manner, and embraces in one argument both finite and infinite sets.

2. **Some lemmas.** Of the three lemmas established in this section, the first one is of interest apart from the application given it in this paper.

LEMMA 1. *If each set of $n+1$ points of a subset M of E_n is enclosable by $S_{n-1,r}$ of given radius r, then M is itself enclosable by this $S_{n-1,r}$.*

PROOF. Consider the family of spheres with centers at points of M and radius r. Since each set of $n+1$ points of M is enclosable by $S_{n-1,r}$ it is clear that each $n+1$ of these spheres have a point in common. Thus each $n+1$ of a family of convex bodies in E_n have a common point, and it follows from a theorem of Helly that there is a point p common to all the members of the family.[5] Each point of M has, then, a distance from p not exceeding r, and hence M is a subset of the sphere with radius r and center p. Thus, the surface $S_{n-1,r}$ of this sphere encloses M.

This lemma permits the reduction of the problem to a finite one concerning $n+1$ points.

LEMMA 2. *Let $P=(p_1, p_2, \cdots, p_{n+1})$ be a set of $n+1$ points of E_n with diameter $d>0$. There exists a positive number r such that P is enclosable by $S_{n-1,r}$ and not enclosable by any S_{n-1,r^*} with $r^*<r$.*

The proof follows readily from the fact that bounded subsets of the E_n are compact.

We establish now some properties of an $S_{n-1,r}$ of smallest radius enclosing an independent set P of $n+1$ points, $p_1, p_2, \cdots, p_n, p_{n+1}$, of E_n.

---

[4] Thus, for example, the obvious property of containing at least one point of the finite set on its circumference is not necessarily carried over for infinite sets. Again, in the treatment of the plane finite case by Rademacher and Toeplitz (*Von Zahlen und Figuren*, 2d edition, 1933, pp. 83–89), it is shown that no arc of the smallest circle as large as a semicircle can be free from points of the set. This also may be invalid when infinite sets are considered.

[5] E. Helly, *Über Mengen konvexer Körper mit gemeinschaftlichen Punkten*, Jahresbericht der Deutschen Mathematiker-Vereinigung, vol. 32 (1923), pp. 175–176. The use of Helly's theorem to prove Lemma 1, conjectured by one of us, occurs in C. V. Robinson's Missouri thesis.

PROPERTY 1. *The center c of $S_{n-1,r}$ is a point of the simplex whose vertices are the points of P.*

PROOF. If the contrary be assumed, a "face" of the simplex separates c from the vertex opposite this hyperplane. It is at once apparent that the $(n-1)$-dimensional spherical surface S_{n-1,r^*} erected on the intersection of this hyperplane with $S_{n-1,r}$ encloses P and has a radius r^* less than r.[6]

PROPERTY 2. *If a point of P is not on $S_{n-1,r}$, then c lies in the face of the simplex opposite this point.*

PROOF. Assuming the contrary, let p_j be a point of P not on $S_{n-1,r}$, and select a Cartesian coordinate system so that the $(n-1)$-dimensional hyperplane determined by the points $p_1, p_2, \cdots, p_{j-1}, p_{j+1}, \cdots, p_{n+1}$ has equation $x_n=0$ and $x_n^{(j)}>0$. It follows then from Property 1 that c_n, the nth coordinate of c, is positive. Let t be any positive number less than the smaller of the numbers $|S_{n-1,r}(p_j)|/2x_n^{(j)}$, c_n, and consider the $(n-1)$-dimensional spherical surface S_{n-1,r^*} with equation

$$S_{n-1,r} + 2tx_n = 0.$$

It is clear that the left-hand member of this equation is negative or zero for each of the points $p_1, p_2, \cdots, p_{j-1}, p_{j+1}, \cdots, p_{n+1}$ since $S_{n-1,r}$ encloses these points and each of them is in the plane $x_n=0$. But also

$$S_{n-1,r}(p_j) + 2tx_n^{(j)} < S_{n-1,r}(p_j) + |S_{n-1,r}(p_j)| = 0,$$

according to the selection of t, and hence S_{n-1,r^*} encloses all of the $n+1$ points of P. This is impossible, for since S_{n-1,r^*} is a linear combination of $S_{n-1,r}$ and the plane $x_n=0$, it passes through the intersection of these loci, and since $0<t<c_n$ the center $(c_1, c_2, \cdots, c_{n-1}, c_n-t)$ of S_{n-1,r^*} is nearer the plane $x_n=0$ than is the center c of $S_{n-1,r}$. Hence $r^*<r$ and the minimum property of r is contradicted.

Remark 1. It follows from Properties 1 and 2 that if c is an *interior* point of the simplex with vertices $p_1, p_2, \cdots, p_{n+1}$ then $S_{n-1,r}$ is the surface of the sphere circumscribing the simplex.

Remark 2. The surface of the sphere circumscribing a simplex is an $(n-1)$-dimensional spherical surface of smallest radius enclosing the

[6] In a properly selected Cartesian coordinate system the separating hyperplane has equation $x_n=0$, and the nth coordinates c_n and $x_n^{(n+1)}$ of c and p_{n+1} (the opposite vertex), respectively, differ in sign. S_{n-1,r^*} has equation $S_{n-1,r}+2c_nx_n=0$, and hence evidently encloses the $n+1$ points $p_1, p_2, \cdots, p_{n+1}$, while $r^*<r$ since the center of S_{n-1,r^*} is the orthogonal projection of c on $x_n=0$.

vertices of the simplex if and only if the circumcenter is a point of the simplex. Since we shall not use this simple criterion in establishing the general theorem, we omit its proof.

LEMMA 3. *Let P be a set of $n+1$ points of E_n, not of E_{n-1}, of diameter d. If $S_{n-1,r}$ is an $(n-1)$-dimensional spherical surface of smallest radius r enclosing P, then $r \leqq [n/2(n+1)]^{1/2} \cdot d$.*

PROOF. Let $P_1, P_2, \cdots, P_{n+1}$ be vectors corresponding to the $n+1$ points $p_1, p_2, \cdots, p_{n+1}$ of P, respectively, while C denotes the vector corresponding to the center c of $S_{n-1,r}$. From Property 1, c is a point of the simplex whose vertices are the points of P and hence *non-negative* constants $k_1, k_2, \cdots, k_{n+1}$ exist such that

$$C = k_1P_1 + k_2P_2 + \cdots + k_{n+1}P_{n+1}, \tag{1}$$

with $k_1+k_2+\cdots+k_{n+1}=1$, and we may suppose the labeling of the $n+1$ points so that (2) $k_{n+1} \geqq k_i$ $(i=1, 2, \cdots, n+1)$. Then k_{n+1} is surely positive and hence c does not lie in the face of the simplex opposite p_{n+1}. It follows from Property 2 that p_{n+1} is on $S_{n-1,r}$.

Translating the origin of coordinates to p_{n+1} does not change the constants $k_1, k_2, \cdots, k_{n+1}$, and we have the scalar product $P_i \cdot (2C-P_i)$ of the vectors P_i and $2C-P_i$ equal to zero *for each index i such that* $k_i>0$, for if k_i is positive then p_i is on $S_{n-1,r}$. Hence, for each such index i, $2P_i \cdot C = P_i \cdot P_i$, but the equality $2k_iP_i \cdot C = k_i(P_i \cdot P_i)$ evidently holds for *every* $i=1, 2, \cdots, n+1$, since k_i is positive or zero. Summing for $i=1, 2, \cdots, n$, we get

$$\sum_{i=1}^{n} (k_iP_i) \cdot C = \frac{1}{2} \sum_{i=1}^{n} k_i(P_i \cdot P_i) = \frac{1}{2} \sum_{i=1}^{n} k_id_i^2,$$

where d_i denotes the length of P_i. Using (1) and recalling that the diameter of P is d, we conclude (since $P_{n+1}=0$) that $C \cdot C \leqq \frac{1}{2}d^2 \sum_{i=1}^{n} k_i$, from which it follows at once (since $\sum_{i=1}^{n} k_i = 1-k_{n+1}$ and using (2)) that $r^2 \leqq [n/2(n+1)] \cdot d^2$; that is, $r \leqq [n/2(n+1)]^{1/2} \cdot d$, and the lemma is proved.[7]

Apart from the uniqueness (which is proved in the general theorem that follows), the Lemmas 2 and 3 prove our theorem for simplices in E_n, while Properties 1 and 2 and the accompanying remarks give important characteristics of a smallest enclosing $S_{n-1,r}$. Lemma 1 presents the means of extending the proof to any bounded subset of E_n.

[7] The treatment by vectors was suggested by C. Herring as a more elegant presentation than the one originally obtained.

3. **The theorem.** We are now in a position to prove quite easily the principal theorem.

THEOREM. *Let d be the diameter of the bounded set M (containing more than a single point) of the n-dimensional euclidean space E_n. Then* (1) *there exists a unique smallest spherical surface $S_{n-1,r}$ enclosing M and* (2) $r \leqq [n/2(n+1)]^{1/2} \cdot d$.

PROOF. The theorem is obviously valid for $n=1$ (the desired $S_{0,r}$ evidently consists of the endpoints of the closure of M). We make the inductive hypothesis of its validity for every positive integer k less than n.

Case 1. *M is a subset of E_k, $1 \leqq k < n$.* Then by the inductive hypothesis there exists a unique smallest $S_{k-1,r}$ enclosing M and $r \leqq [k/2(k+1)]^{1/2} \cdot d < [n/2(n+1)]^{1/2} \cdot d$ since $k<n$. It is clear that the $S_{n-1,r}$ which is the surface of the n-dimensional sphere whose radius is r and whose center coincides with that of the k-dimensional sphere with surface $S_{k-1,r}$ satisfies the requirements of the theorem.

Case 2. *M is not a subset of E_k, $k<n$.* Then the set $\{P\}$ of all sets of $n+1$ points of M is not empty, and by Lemma 2 there is a smallest $S_{n-1,r(P)}$ enclosing each P of $\{P\}$. Define $r = \text{l.u.b.}_{P \in \{P\}} r(P)$. Since $0 < r(P) < d$, $P \in \{P\}$, r is a positive (finite) number.

ASSERTION. *M is enclosable by $S_{n-1,r}$ and by no spherical surface of smaller radius.*

First, since $r \geqq r(P)$, $P \subset M$, it follows that each set of $n+1$ points of M is enclosable by $S_{n-1,r}$ and hence, by Lemma 1, M is itself enclosable by $S_{n-1,r}$. Second, an assumption that M is enclosable by S_{n-1,r^*}, $r^* < r$, leads at once (according to the definition of r) to the existence of a subset P of $n+1$ points of M with $r(P) > r^*$; that is, the smallest spherical surface enclosing this subset P has a radius exceeding r^*. Hence this subset (and consequently M) is not enclosable by an S_{n-1,r^*}.

Let $S_{n-1,r}(p)$ denote an $(n-1)$-dimensional spherical surface of smallest radius r enclosing M with center p. If, now, $S_{n-1,r}(q)$ is another such spherical surface, then M is contained in the common part of the two n-dimensional spheres of radius r and centers p and q, and consequently is part of an n-dimensional sphere of radius $r^* < r$. Then M is enclosable by S_{n-1,r^*}, $r^* < r$, which is impossible. Hence the center as well as the radius of $S_{n-1,r}$ is fixed and the uniqueness is established.

Finally, $r(P) \leqq [n/2(n+1)]^{1/2} \cdot d$ for each P of $\{P\}$. For if P is in E_k, $k<n$, this follows from Case 1, while if P consists of $n+1$

independent points then the result follows from Lemma 3. Hence $r = \text{l.u.b.}_{P \in [P]} r(P) \leqq [n/2(n+1)]^{1/2} \cdot d$, and the proof of the theorem is complete.

If $p_1, p_2, \cdots, p_{n+1}$ are the vertices of an equilateral simplex of edge d then the unique $(n-1)$-dimensional spherical surface of smallest radius enclosing the points is, according to Remark 2, the surface of the circumscribed sphere. Since for this sphere $r = [n/2(n+1)]^{1/2} \cdot d$, the inequality proved for r in the theorem cannot be sharpened.

4. **Some historical remarks.** Though Jung does not mention the fact in his dissertation, some aspects of the problem he dealt with had been considered long before. Thus Sylvester, in a paper on approximate valuation of surd forms, asserted that the essential preliminary question to be resolved was that of cutting off by a plane the smallest possible segment of a sphere that should contain the whole of a given (finite) set of points lying on the surface.[8] He stated that some years earlier he had proposed the problem of drawing the smallest circle enclosing a given finite set of points in the plane "without any suspicion of its having a practical application," and that "by a singular coincidence, Professor Peirce, of Cambridge University (*sic*), U.S., has studied this question and obtained a complete solution" which was applicable to the analogous problem on the sphere.

Unaware of either Sylvester's or Peirce's connection with the problem, G. Chrystal (1884) gave a geometrical algorithm for drawing a smallest circle enclosing a given finite set of points in the plane.[9] No mention is made in any of this earlier work of a relationship between the diameter of the set and the radius of the smallest circle containing it.

In 1914 M. Bricard, apparently unaware of either of Jung's two earlier papers, proved that every closed plane curve of elongation (diameter) d is contained in a circle of radius $d/3^{1/2}$ and every closed surface of diameter d is contained in a sphere of radius $(3/8)^{1/2}d$.[10] In a brief note H. Lebesgue remarked that Bricard's methods might be applied to any plane set E of diameter d by considering E as a subset of the set E^* which is *saturated with respect to having diameter*

---

[8] J. J. Sylvester, *On Poncelet's approximate valuation of surd forms*, Philosophical Magazine, vol. 20 (1860), pp. 203–222; see also *Collected works*, vol. 2, pp. 118, 183.

[9] G. Chrystal, *On the problem to construct the minimum circle enclosing n given points in a plane*, Proceedings of the Edinburg Mathematical Society, vol. 3 (1884–1885), pp. 30–33.

[10] M. Bricard, *Théorèmes sur les courbes et les surfaces fermées*, Nouvelles Annales de Mathématiques, (4), vol. 14 (1914), pp. 19–25.

d.[11] Such sets E^* are convex domains bounded by curves that Euler called *orbiform* (that is, curves of constant breadth).

A kind of dual of Jung's theorem was proved by W. Blaschke (1914).[12] If E is a plane convex set, the breadth of E is defined as the minimum distance of two parallel supporting lines of E. Blaschke showed that *the greatest circle which is contained in every plane convex set of breadth* 1 *has diameter* 2/3, and established analogous results for higher dimensions. It was pointed out by J. v. Sz. Nagy that Blaschke's methods could be applied to prove Jung's theorem.[13] In 1917 K. Reinhardt sought to extend to the E_n the geometrical arguments used by Jung in 1909 for subsets of E_2.[14] The demonstration is tedious and not entirely non-intuitive. An analogue of Jung's theorem in more general spaces was proved in 1938 by F. Bohnenblust.[15]

Finally, we note that in 1905 E. Landau applied Jung's theorem in the plane to sharpen an inequality in the theory of analytic functions due to F. Schottky.[16]

UNIVERSITY OF MISSOURI

[11] H. Lebesgue, *Sur les courbes orbiformes, à propos d'une note récente de M. R. Bricard*, Bulletin de la Société Mathématique de France, Comptes Rendus des Séances, 1914, pp. 45–46 (abstract).

[12] W. Blaschke, *Über den grössten Kreis in einer konvexen Punktmenge*, Jahresbericht der Deutschen Mathematiker-Vereinigung, vol. 23 (1914), pp. 369–374.

[13] J. v. Sz. Nagy, *Über einen Satz von H. Jung*, ibid., vol. 24 (1915), pp. 390–392

[14] K. Reinhardt, *Über die kleinste Kugel, die um jede Punktmenge vom Durchmesser Eins gelegt werden kann*, ibid., vol. 25 (1917), pp. 157–163.

[15] F. Bohnenblust, *Convex regions and projections in Minkowski spaces*, Annals of Mathematics, (2), vol. 39 (1938), pp. 301–308.

[16] E. Landau, *Über einige Ungleichheitsbeziehungen in der Theorie der analytischen Funktionen*, Archiv der Mathematik und Physik, (3), vol. 11 (1905), pp. 31–36.

THE ACYCLIC ELEMENTS OF A PEANO SPACE

A. D. WALLACE

We suppose throughout that S is a Peano space. The notion of a cyclic element of such a space was introduced by G. T. Whyburn[1] and A-sets were introduced independently by W. L. Ayres[2] and Whyburn. Our present purpose is to consider a class of sets which may be regarded as the duals of cyclic elements.

Let $Q(S)$ denote the set of all cut-points and end-points of S.[3] Then each component of $Q(S)$ will be called an *acyclic element* of S. If p and q are two points of S then we write $p \sim q$ to mean that *no point separates p from q in S*. A set will be termed *acyclic* if it contains no simple closed curve, *cyclic* if each pair of points is on a simple closed curve of the set.

(i) *Each cyclic element* [*acyclic element*] *is a cyclic* [*acyclic*] A*-set.*

PROOF. We prove only the second statement. Let F be an acyclic element, $x_n \in F$ and suppose that $x_n \to x$. If x is not in F then since $F+x$ is connected we must have $x \in S - Q(S)$, from the definition of F as a component of $Q(S)$. Consequently x is a point of a cyclic element E of S. If $(F+x) \cdot E$ contained only the point x then it would follow that F was contained in a single complementary domain R of E and thus $x = F(R) = \overline{R} - R$. Consequently x would be a cut-point, which is obviously impossible. We conclude that $(F+x) \cdot E$ is a nondegenerate connected set. It is clear that no point of this set is an end-point and hence the set contains uncountably many cut-points. But no cyclic element contains more than a countable number of cut-points. We conclude that $x \in F$ and hence F is closed. If F is not an A-set there is an arc pxq which meets F in the set $p+q$. It is manifest that no point separates p from q in S since F is connected. That is, $p \sim q$ and hence $p+q \subset E$, a true cyclic element. By the argument given

[1] See Kuratowski and Whyburn, Fundamenta Mathematicae, vol. 16 (1930), p. 305. By a cyclic element we understand a *nondegenerate* set such that any two points lie on a simple closed curve and which is maximal relative to this property. This is not the definition given by Kuratowski and Whyburn but is equivalent (cf. G. T. Whyburn, this Bulletin, vol. 38 (1931), p. 429 and references given there) and is in fact Whyburn's original definition. It seems more natural in the present setting.

[2] See W. L. Ayres, this Bulletin, vol. 46 (1940), p. 794, for references. An A-set is a *closed* arc set.

[3] The terms *cut-point* and *end-point* without qualification refer to the space S. The proofs of all the statements concerning cyclic elements will be found in Kuratowski and Whyburn. We assume considerable familiarity with this fundamental work.

above E would contain uncountably many cut-points, a contradiction. Thus F is an A-set. Since each A-set is a Peano space it is easy to see that F contains no simple closed curve.

(ii) *In order that a set X of S be a cyclic element [acyclic element] it is necessary and sufficient that it be a connected set which is maximal relative to the property: If p and q are points of X then $p \sim q$ [p non$\sim q$].*

PROOF. We prove only the second statement. Let F be an acyclic element of S, $p, q \in F$, and let t be the unique arc from p to q in F. Since F is a tree there is a point x on t which separates p from q in F. But since F is an A-set we know that x separates p from q in S so that we have p non$\sim q$. Let F' be a connected set which contains F and which further contains at least one point which does not belong to F. Because F' is connected it contains a point p of $S-Q(S)$ and hence a point of E, a cyclic element. It is easy to see that if F' and E have in common only the point p then p must be a cut-point, which is impossible since $p \in S-Q(S)$. Hence $F' \cdot E$ contains another point q. But since $p, q \in E$ we have $p \sim q$ and by assumption p non$\sim q$ because $p, q \in F'$. Thus $F = F'$. The condition of the theorem is thus necessary. It is not hard to see that the condition is sufficient.

(iii) *No point of a cyclic element E is an end-point or a cut-point of E while every point of an acyclic element F is either an end-point or a cut-point of F.*

We use the term *element* indiscriminately for cyclic or acyclic element.

(iv) *The elements of S form a null sequence in the sense that for any positive number δ there are only finitely many whose diameters exceed δ.*

PROOF. The result is known for cyclic elements. If there existed an infinite sequence of acyclic elements with diameters bounded from zero we could find an infinite subsequence with the same property and which converged to a nondegenerate continuum, that is, we have $F_n \to X$, $\delta(X) \geq d > 0$. If $a, b \in X$ and a non$\sim b$, then we may write $S = M + N$, where $a \in M$, $b \in N$, $M \cdot N = p$, a cut-point. For all large n it follows that F_n meets both M and N and hence $p \in F_n$, contrary to the fact that $F_i \cdot F_j = 0$ for $i \neq j$. We conclude that $X \subset E$, a cyclic element. It is easy to see that we can find two points $u, v \in X - \sum F_n$, since each acyclic element meets E in at most one point and the set X is uncountable. For n sufficiently large we can find arcs uu' and vv' which are disjoint and such that $F_n \cdot uu' = u'$, $F_n \cdot vv' = v'$. Since F_n is an A-set we can find an arc $u'v'$ in F_n. Further we can find an arc uv

in E. The set $uv+u'v'+uu'+vv'$ contains a simple closed curve J which has two points in F_n. Since we may suppose that J meets E in $u+v$ we must have $J \subset E$ and thus F_n meets E in more than one point.

(v) *A Peano space is the sum of its elements and no two have more than one point in common. No two acyclic elements meet.*

(vi) *The decomposition of S into acyclic elements and points of $S-Q(S)$ is upper semi-continuous. Thus the associated monotone transformation $TS=S'$ of S onto the hyper-space S' is topological on each cyclic element of S and the acyclic elements of S' are all degenerate.*

The results stated do not exhaust the list of dual properties but we have tried to give the most striking. It appears improbable that the idea could be extended to non-locally connected spaces using the definitions given by Kelley,[4] Moore,[5] or Whyburn;[6] or to the generalizations due to Hall[7] and Youngs.[8]

THE UNIVERSITY OF VIRGINIA AND
PRINCETON UNIVERSITY

[4] J. L. Kelley, Proceedings of the National Academy of Sciences, vol. 26 (1940), p. 192.

[5] R. L. Moore, *Foundations of Point Set Theory*, American Mathematical Society Colloquium Publications, vol. 13, New York, 1932.

[6] G. T. Whyburn, American Journal of Mathematics, vol. 56 (1934), p. 133.

[7] D. W. Hall, Transactions of this Society, vol. 47 (1940), p. 305.

[8] J. W. T. Youngs, American Journal of Mathematics, vol. 52 (1940), p. 449.

ON CERTAIN BASIC SERIES

JOHN A. DAUM

1. **Introduction.** The identity

$$(1)\quad \sum_{n=1}^{\infty} \frac{q^n}{(1-q^n)^2}\left\{\frac{1}{1-q}+\frac{1}{1-q^2}+\cdots+\frac{1}{1-q^n}\right\} = \sum_{n=1}^{\infty}\frac{n^2q^n}{1-q^n}$$

was deduced from arithmetical considerations by E. T. Bell. About five years ago, W. N. Bailey[1] proved the relation

$$(2)\quad \sum_{n=0}^{\infty} \frac{(1-q)(1-q^2)\cdots(1-q^n)}{(1-z)(1-qz)\cdots(1-q^nz)}\cdot\frac{z^{n+1}}{1-q^{n+1}} = \sum_{n=0}^{\infty}\frac{q^nz}{(1-q^nz)^2},$$

from which he obtained (1) by differentiating with respect to z and then putting $z=q$. A short time later Hall[2] gave an alternate proof of (2) by simply specializing the parameters in a relation between basic series.[3]

Since Bell's identity (1) leads to interesting arithmetical results, it would be desirable to obtain as many more similar relations as possible. Hall's paper, as well as Bailey's, seems to indicate a connection with certain formulae in basic series. The identity which Hall employed to obtain Bailey's relation is the basic analogue of but one of many identities involving the series ${}_3F_2$ which were proved originally by Thomae[4] and later classified by Whipple.[5]

The object of this paper is to investigate the possibility of systematically obtaining new results similar to Bailey's by employing the complete set of basic analogues of Thomae's two-term transformations.

2. **The basic analogues of Thomae's relations.** We first establish the complete set of basic analogues of Thomae's relations, using a notation analogous to that used by Whipple.

Let $I=(a, b, c, d, e, f)$ be a matrix whose elements form an arbitrary permutation of the integers 0, 1, 2, 3, 4, 5 and let $r_a, r_b, r_c, r_d, r_e, r_f$ be six numbers such that their product is unity. Define parameters A and B by

[1] W. N. Bailey, *An algebraic identity*, Journal of the London Mathematical Society, vol. 11 (1936), p. 156.

[2] N. A. Hall, *An algebraic identity*, ibid., vol. 11 (1936), p. 276.

[3] W. N. Bailey, *Generalized Hypergeometric Series*, Cambridge Tract, no. 32, p. 65.

[4] J. Thomae, Journal für die reine und angewandte Mathematik, vol. 87 (1879), p. 26.

[5] F. J. Whipple, Proceedings of the London Mathematical Society, vol. 23 (1925), p. 104.

$$A_{uvw} = r_u r_v r_w q^{1/2}, \quad |q| < 1; \qquad B_{uv} = r_u q / r_v;$$

and functions Φp and Φn by

$$\Phi p(a; b, c) = \frac{1}{\Omega(A_{def})\Omega(B_{ba})\Omega(B_{ca})} \; {}_3\Phi_2\left[\begin{matrix} A_{dbc}, & A_{ebc}, & A_{fbc}; & A_{def} \\ & B_{ba}, & B_{ca} & \end{matrix}\right],$$

$$\Phi n(a; b, c) = \frac{1}{\Omega(A_{abc})\Omega(B_{ab})\Omega(B_{ac})} \; {}_3\Phi_2\left[\begin{matrix} A_{adc}, & A_{adf}, & A_{aef}; & A_{abc} \\ & B_{ab}, & B_{ac} & \end{matrix}\right],$$

where[6]

$${}_3\Phi_2\left[\begin{matrix} \alpha, & \beta, & \gamma; & z \\ & \delta, & \epsilon & \end{matrix}\right] = \frac{\Omega(\delta)\Omega(\epsilon)}{\Omega(\alpha)\Omega(\beta)\Omega(\gamma)} \sum_{n=0}^{\infty} \frac{\Omega(\alpha q^n)\Omega(\beta q^n)\Omega(\gamma q^n)}{\Omega(q^{n+1})\Omega(\delta q^n)\Omega(\epsilon q^n)} z^n, \quad |z| < 1,$$

and[7]

$$\Omega(x) = \prod_{n=0}^{\infty} \frac{1 - q^{n+1}}{1 - xq^n}.$$

Considering all possible matrices I, it is possible to form 60 Φp's and 60 Φn's.

There are 90 fundamental relations between certain pairs of the functions Φp. The proof of these relations, which can all be written in the form

$$\Phi p(a; b, c) = \Phi p(a; d, e), \tag{3}$$

is almost identical with that of the original case as given by Bailey,[8] subject to certain analogous conditions. The only essential difference is in the replacement of the gamma-function throughout by the modified Heine omega-function.

The 90 identities (3) are all equivalent in the sense that any one of these identities may be obtained directly from any other by proper determination of the r's. The corresponding relations involving the functions Φn are all equivalent to those involving the Φp's—one set being obtainable from the other by replacing the r's by their reciprocals.

As an example of the use of relations (3), consider the identity

$$\Phi p(0; 4, 5) = \Phi p(0; 2, 3),$$

where

[6] W. N. Bailey, *Generalized Hypergeometric Series*, Cambridge Tract, no. 32, p. 65.
[7] N. A. Hall, loc. cit.
[8] W. N. Bailey, *Generalized Hypergeometric Series*, Cambridge Tract, no. 32, p. 14.

$A_{145} = \alpha, \quad A_{245} = \beta, \quad A_{345} = \gamma, \quad B_{40} = \delta, \quad B_{50} = \epsilon, \quad A_{123} = \delta\epsilon/\alpha\beta\gamma.$

We obtain the identity

$$ {}_3\Phi_2\left[\begin{matrix}\alpha, \beta, \gamma; \delta\epsilon/\alpha\beta\gamma \\ \delta, \epsilon\end{matrix}\right] = \frac{\Omega(\delta)\Omega(\epsilon)\Omega(\delta\epsilon/\alpha\beta\gamma)}{\Omega(\alpha)\Omega(\delta\epsilon/\alpha\beta)\Omega(\delta\epsilon/\alpha\gamma)} \cdot {}_3\Phi_2\left[\begin{matrix}\delta/\alpha, \epsilon/\alpha, \delta\epsilon/\alpha\beta\gamma; \alpha \\ \delta\epsilon/\alpha\beta, \delta\epsilon/\alpha\gamma\end{matrix}\right], $$

which was used by Hall to obtain Bailey's identity (2).

For a fixed value of a (3) represents 15 identities between certain of the functions $\Phi p(a)$. For $a=0$, these identities are

$$ \begin{aligned} \Phi p(0;4,5) &= \Phi p(0;2,3) & \Phi p(0;1,5) &= \Phi p(0;2,3) \\ &= \Phi p(0;1,3) & &= \Phi p(0;2,4) \\ &= \Phi p(0;1,2), & &= \Phi p(0;3,4), \\ \Phi p(0;3,5) &= \Phi p(0;1,2) & \Phi p(0;1,4) &= \Phi p(0;2,3) \\ &= \Phi p(0;1,4) & \Phi p(0;2,4) &= \Phi p(0;1,3) \\ &= \Phi p(0;2,4), & \Phi p(0;3,4) &= \Phi p(0;1,2), \\ \Phi p(0;2,5) &= \Phi p(0;1,3) \\ &= \Phi p(0;1,4) \\ &= \Phi p(0;3,4). \end{aligned} \tag{4} $$

From this, it is clear that we have a second set of identities of the form

$$ \Phi p(a; b, c) = \Phi p(a; b, d) \tag{5} $$

subject to certain conditions obtained from those applied in proving (3). Relation (5) represents 180 equivalent identities. Typical of results obtained are

$$ {}_3\Phi_2\left[\begin{matrix}\alpha, \beta, \gamma; \delta\epsilon/\alpha\beta\gamma \\ \delta, \epsilon\end{matrix}\right] = \frac{\Omega(\delta)\Omega(\epsilon)\Omega(\delta\epsilon/\alpha\beta\gamma)}{\Omega(\delta\epsilon/\alpha\beta)\Omega(\epsilon)\Omega(\delta/\gamma)} {}_3\Phi_2\left[\begin{matrix}\epsilon/\beta, \epsilon/\alpha, \gamma; \delta/\gamma \\ \delta\epsilon/\alpha\beta, \epsilon\end{matrix}\right] \tag{6} $$

and

$$ {}_3\Phi_2\left[\begin{matrix}\delta\epsilon/\alpha\beta\gamma, \delta/\alpha, \epsilon/\alpha; \alpha \\ \delta\epsilon/\alpha\gamma, \delta\epsilon/\alpha\beta\end{matrix}\right] = \frac{\Omega(\delta\epsilon/\alpha\gamma)\Omega(\delta\epsilon/\alpha\beta)\Omega(\alpha)}{\Omega(\delta\epsilon/\beta\gamma)\Omega(\delta\epsilon/\alpha\beta)\Omega(\beta)} \cdot {}_3\Phi_2\left[\begin{matrix}\delta\epsilon/\alpha\beta\gamma; \delta/\beta, \epsilon/\beta; \beta \\ \delta\epsilon/\beta\gamma, \delta\epsilon/\alpha\beta\end{matrix}\right]. \tag{7} $$

It is possible to obtain Bailey's identity from either (6) or (7).

3. **Conclusion.** This classification of the basic analogues of Thomae's two-term relations indicates, then, that out of the entire set of identities, only two are essentially distinct. Any further results can be obtained from this source only by specialization of the parameters in Hall's identity or in relation (6) or (7).

Analogous statements can be made concerning the corresponding relations between the generalized hypergeometric functions ${}_3F_2$ as given by Whipple.

AGRICULTURAL AND MECHANICAL COLLEGE OF TEXAS AND
UNIVERSITY OF NEBRASKA

A NOTE ON FINITE REGULAR RINGS

JOHN DYER-BENNET

The purpose of this note is to give a classification of all finite regular rings. One result of the classification is the extension of a part of Wedderburn's classic theory of the structure of linear associative algebras to systems which are not algebras. I wish to thank Professor Garrett Birkhoff for several suggestions in the preparation of the note.

DEFINITION. *A ring R is said to be regular if for every element a of R there exists an element x in R such that*

$$aka = a. \tag{1}$$

[1]

LEMMA 1. *A regular ring has radical* (0).[2]

PROOF. Since the radical of a ring is a nilpotent ideal, any element a in it has the property that a product containing a as a factor a sufficient number of times is zero. Since by definition there exists an x such that $axax \cdots axa = a$ no matter how many terms there are in the product, zero must be the only member of the radical.

[1] Cf. von Neumann, *Lectures on Continuous Geometries*, Part 2, Princeton, 1937, pp. 7–21. Also *On regular rings*, Proceedings of the National Academy of Sciences, vol. 22 (1936), pp. 707–713. von Neumann postulates the existence of a unit, but in our case this follows from the other assumptions.

[2] Cf. von Neumann, loc. cit.

From Lemma 1 it follows that a regular ring which satisfies the finite descending chain condition for ideals is semi-simple, and hence contains a unit.

DEFINITION. *The characteristic of a ring is the least common multiple of the additive orders of its elements, if this is finite, otherwise* ∞.

If the ring has a unit this is the additive order of the unit, so that this definition agrees with the usual one for a field.

LEMMA 2. *The characteristic of a regular ring with a unit is square-free, if it is finite.*

PROOF. Suppose the characteristic c is not square-free. Let

$$c = p_1^{e_1} p_2^{e_2} \cdots p_r^{e_r}$$

where we may assume that $e_1 > 1$. Let $a = c/p_1$ where we are considering a as an element of the ring. Then $a \neq 0$. Further, for any x in the ring

$$axa = aax = 0 \neq a.$$

It follows that the ring is not regular.

THEOREM 1. *A finite regular ring is the direct sum of algebras.*

PROOF. By the theory of commutative groups the additive group of the ring is the direct sum of subgroups each of which consists of all the elements whose order is a fixed prime p.[3] Since the order of a product of two elements of order p is also of order p—unless the product is zero—these subgroups are also subrings. Next, let a and b be two elements of additive orders p and q respectively, $p \neq q$. Then

$$pab = (pa)b = 0 \cdot b = 0$$

and

$$qab = a(qb) = a \cdot 0 = 0,$$

where pc stands for a sum of p summands c. Thus the order of ab is less than or equal to p and to q, and divides both. Hence $ab = 0$. Since the product of any two elements from different subrings is zero, the ring is the direct sum of these subrings. Each of these subrings is of prime characteristic, and hence contains in its center a prime field of p elements. Over this field it is clearly a linear associative algebra. Finally, each of these is clearly semi-simple.

[3] Cf. Speiser, *Theorie der Gruppen von endlicher Ordnung*, 3d edition, p. 49.

THEOREM 2. *A finite regular ring is the direct sum of full matrix algebras over finite fields, and the summands are unique, apart from their order.*

PROOF. By Theorem 1 it is the direct sum of semi-simple algebras. By a theorem due to Wedderburn each of these is a direct sum of simple algebras, and these are unique, apart from their order.[4] But a simple algebra is a total matrix algebra over a division algebra.[5] Finally, every finite division algebra is commutative, and hence a field.[6]

The proof of Theorem 2 above seems to lend itself to generalization, but it is not the simplest proof that can be given. By relying even more heavily on results in the literature the theorem can be proved as follows.

von Neumann has shown that a regular ring with a unit is irreducible if and only if its center is a field.[7] Hence a finite regular ring is the direct sum of (finite regular) rings whose centers are fields. The terms in this direct sum are uniquely determined.[8] This reduces the problem to that of classifying the finite regular rings whose centers are fields. Such a ring is a normal simple linear associative algebra over its center,[9] therefore a total matric algebra over its center.[10]

There is an immediate corollary of the results above. Since a regular ring is irreducible if and only if its center is a field, there follows the theorem (due to Dedekind[11]) that a commutative regular ring satisfying the descending chain condition is the direct sum of fields.

We can also get another proof of what is almost Lemma 2, namely that a semi-simple ring has a square-free characteristic if the characteristic is finite. For the direct sum of two fields of characteristic p is of characteristic p, while the direct sum of two fields of characteristics p and q respectively has characteristic pq. This result is interesting for the following reason. The regularity condition (1) above is a natural weakening of the condition that inverses should exist,[12] and so a regular ring might be thought of as somewhere between an arbi-

---

[4] Albert, *Structure of Algebras*, American Mathematical Society Colloquium Publications, vol. 24, p. 39.

[5] Cf. Albert, loc. cit., p. 39.

[6] Cf. Albert, loc. cit., p. 62.

[7] *Continuous Geometries*, Part 2, p. 21.

[8] van der Waerden, *Moderne Algebra*, vol. 2, p. 162.

[9] Albert, loc. cit., pp. 6, 37.

[10] Albert, loc. cit., p. 62.

[11] Cf. van der Waerden, loc. cit., p. 163.

[12] Cf. von Neumann, *Continuous Geometries*, Part 2, p. 21.

trary ring and a field. Similarly the result that the characteristic is square-free is a natural weakening of the result that the characteristic of a finite field is prime.

We complete the note by giving a combinatory formula for the number of non-isomorphic regular rings of a given finite order. The formula is unsatisfactory in that it involves finding all the partitions of an integer. It does not seem likely, however, that this can be avoided.

The algebra of n^2 matrices over a field of p^k elements has p^{n^2k} elements. Hence there are as many total matrix algebras with p^e elements as there are squares that divide e. Let

$$e = p_1^{e_1} \cdots p_r^{e_r}.$$

Then there are clearly

$$\prod_{i=1}^{r} \{[e_i/2] + 1\}$$

such algebras, where $[a]$ is the greatest integer in a. These are all different, since the center of each is isomorphic with the field over which it is taken.

The number of non-isomorphic regular rings, each of which is the direct sum of s simple rings of orders p^{e_i} respectively is the product of the numbers for each of the e_i. The number of order p^k is therefore the sum of these products for all possible partitions of k, $k = \sum_{i=1}^{s} e_i$.

Using these results and Theorem 2 we see that the number of non-isomorphic rings of order $n = \prod_{i=1}^{r} p_i^{k_i}$ is

$$\prod_{i=1}^{r} \sum_{e_1+\cdots+e_s=k_i} \prod_{i=1}^{s} \prod_{j=1}^{r_i} \{[e_{i,j}/2] + 1\}$$

where the summation is taken over all partitions of k_i, and

$$e_i = \prod_{j=1}^{r_i} p_{i,j}^{e_{i,j}}.$$

VANDERBILT UNIVERSITY

OSCULATING QUADRICS OF RULED SURFACES IN RECIPROCAL RECTILINEAR CONGRUENCES

M. L. MACQUEEN

1. **Introduction.** Let x be a general point of an analytic non-ruled surface S referred to its asymptotic net in ordinary projective space. By a line l_1 at the point x we mean any line through the point x and not lying in the tangent plane of the surface at the point x. Dually, a line l_2 is any line in the tangent plane of the surface at the point x but not passing through the point x. The lines l_1, l_2 are called reciprocal lines if they are reciprocal polar lines with respect to the quadric of Lie at the point x. In this case, when the point x varies over the surface S, the lines l_1, l_2 generate two rectilinear congruences Γ_1, Γ_2 which are said to be reciprocal with respect to the surface. If, however, the point x moves along the u-curve, the locus of the line l_1 is a ruled surface $R_1^{(u)}$ of the congruence Γ_1. The osculating quadric along a generator l_1 of the ruled surface $R_1^{(u)}$ is the limit of the quadric determined by the line l_1 through the point x and the lines l_1 through two neighboring points P_1, P_2 on the u-curve as each of these points independently approaches the point x along the u-curve. The quadric thus defined will be denoted by $Q_1^{(u)}$. A second quadric $Q_1^{(v)}$ is determined by three consecutive lines l_1 at points of the v-curve through the point x. Moreover, there are two quadrics, denoted by $Q_2^{(u)}$ and $Q_2^{(v)}$, which are associated with two ruled surfaces of the reciprocal congruence Γ_2 and which can be defined similarly. This note will study the projective differential geometry of the quadrics thus defined.

2. **Analytic basis.** Let the surface S under consideration be an analytic non-ruled surface whose parametric vector equation, referred to asymptotic parameters u, v, is

$$x = x(u, v). \tag{1}$$

The four coordinates x of a variable point x on the surface satisfy two partial differential equations which can be reduced, by a suitably chosen transformation of proportionality factor, to Fubini's canonical form

$$x_{uu} = px + \theta_u x_u + \beta x_v, \qquad x_{vv} = qx + \gamma x_u + \theta_v x_v, \qquad \theta = \log \beta\gamma, \tag{2}$$

in which the subscripts indicate partial differentiation. The coefficients of these equations are functions of u, v and satisfy three integrability conditions which need not be written here.

Two lines l_1, l_2 are reciprocal lines if the line l_1 joins the point x and the point y defined[1] by

$$y = -ax_u - bx_v + x_{uv} \tag{3}$$

and the line l_2 joins the points ρ, σ defined by placing

$$\rho = x_u - bx, \qquad \sigma = x_v - ax, \tag{4}$$

where a, b are functions of u, v. As u, v vary, the lines l_1, l_2 generate two rectilinear congruences Γ_1, Γ_2 which are reciprocal with respect to the surface.

The curves corresponding to the developables of the congruence Γ_1 are called the Γ_1-curves of the congruence, and those corresponding to the developables of the congruence Γ_2 the Γ_2-curves of the congruence. The differential equation of the Γ_1-curves is

$$(F - 2a\beta + \beta\psi)du^2 - (b_v - a_u)dudv - (G - 2b\gamma + \gamma\phi)dv^2 = 0, \tag{5}$$

where F, G are defined by the formulas

$$F = p - b_u + b\theta_u - b^2 + a\beta, \qquad G = q - a_v + a\theta_v - a^2 + b\gamma,$$

and ϕ, ψ by

$$\phi = (\log \beta\gamma^2)_u, \qquad \psi = (\log \beta^2\gamma)_v.$$

If k_1, k_2 are the roots of the equation

$$k^2 + (A + B)k + AB - (F - 2a\beta + \beta\psi)(G - 2b\gamma + \gamma\phi) = 0, \tag{6}$$

where the functions A, B are defined by

$$A = -a_u - ab + \beta\gamma + \theta_{uv}, \qquad B = -b_v - ab + \beta\gamma + \theta_{uv},$$

the corresponding points

$$y + k_i x, \qquad i = 1, 2,$$

are the focal points of the line l_1. Furthermore, the differential equation of the Γ_2-curves is

$$Fdu^2 - (b_v - a_u)dudv - Gdv^2 = 0. \tag{7}$$

If τ_1, τ_2 are the roots of the equation

$$F + (b_v - a_u)\tau - G\tau^2 = 0, \tag{8}$$

the corresponding points

$$\rho + \tau_i\sigma, \qquad i = 1, 2,$$

[1] In this section we employ the notation used by E. P. Lane in Chapter III of his book *Projective Differential Geometry of Curves and Surfaces*, Chicago, 1932.

are the focal points of the line l_2. It will be assumed that the coefficients of du^2 and dv^2 in equations (5), (7) are all nonzero. In this case the Γ_1-curves and the Γ_2-curves of two reciprocal congruences form conjugate nets if, and only if, $b_v - a_u = 0$.

3. **Osculating quadrics of ruled surfaces of the congruence Γ_1.** Any point z, except the point y, on the line l_1 at the point x is given by the equation

$$z = x + \omega y, \qquad \omega \text{ scalar.} \tag{9}$$

As the point x varies along the u-curve, the line l_1 generates a ruled surface $R_1^{(u)}$. Equation (9) is the parametric vector equation of this ruled surface, u, ω being the independent parameters. The asymptotic curves on $R_1^{(u)}$ consist of the lines l_1 and the integral curves of the differential equation

$$L_1 du + 2M_1 d\omega = 0, \tag{10}$$

where L_1, M_1 are determinants of the fourth order defined by

$$L_1 = (z_{uu}, z, z_u, z_\omega), \qquad M_1 = (z_{u\omega}, z, z_u, z_\omega).$$

Differentiating equation (9) and using equations (2), (3) to calculate the values of L_1, M_1, we find that equation (10) can be written in the form

$$\frac{d\omega}{du} = -\frac{\beta + C\omega + D\omega^2}{2(F - 2a\beta + b\psi)}, \tag{11}$$

where we have placed

$$C = F_u - 2(a\beta)_u + (b\psi)_u + 2\beta A,$$
$$D = \beta A^2 - a(F - 2a\beta + b\psi)^2 + A(F - 2a\beta + b\psi)_u$$
$$- (F - 2a\beta + b\psi)[p_v + \beta q - ap + A(b - \theta_u) + A_u].$$

Any point X, except the point z, on the tangent at the point z of the curved asymptotic on $R_1^{(u)}$ is defined by placing

$$X = \lambda z + dz/du, \qquad \lambda \text{ scalar.} \tag{12}$$

If we use the tetrahedron x, ρ, σ, y as a local tetrahedron of reference with a unit point chosen so that a point

$$x_1 x + x_2 \rho + x_3 \sigma + x_4 y$$

has local coordinates proportional to $x_1, \cdots, x_4$, we find that the local coordinates of the point X are given by

$$
\begin{aligned}
x_1 &= b + \lambda + [a(F - 2a\beta + \beta\psi) + bA + p_v + \beta q - ap]\omega, \\
x_2 &= 1 + A\omega, \\
x_3 &= (F - 2a\beta + \beta\psi)\omega, \\
x_4 &= \omega\lambda - (b - \theta_u)\omega - \frac{\beta + C\omega + D\omega^2}{2(F - 2a\beta + \beta\psi)}.
\end{aligned} \tag{13}
$$

Homogeneous elimination of ω, λ from these equations gives *the algebraic equation of the quadric* $Q_1^{(u)}$, *referred to the tetrahedron* x, ρ, σ, y, namely

$$
\begin{aligned}
&\beta(F - 2a\beta + \beta\psi)x_2^2 + Hx_3^2 + 2(F - 2a\beta + \beta\psi)^2 x_2x_4 \\
&\quad - 2A(F - 2a\beta + \beta\psi)x_3x_4 \\
&\quad - 2(F - 2a\beta + \beta\psi)x_1x_2 + 2Px_2x_3 = 0,
\end{aligned} \tag{14}
$$

where the coefficients H, P are defined by

$$
\begin{aligned}
H &= a(F - 2a\beta + \beta\psi) - 3A(b - \theta_u) - A_u + p_v + \beta q - ap, \\
P &= (2b - \theta_u)(F - 2a\beta + \beta\psi) + \tfrac{1}{2}(F - 2a\beta + \beta\psi)_u.
\end{aligned} \tag{15}
$$

The equation of the quadric $Q_1^{(v)}$ can be written by interchanging u and v and making the appropriate symmetrical interchanges of the other symbols. The result is

$$
\begin{aligned}
&K x_2^2 + \gamma(G - 2b\gamma + \gamma\phi)x_3^2 + 2(G - 2b\gamma + \gamma\phi)^2 x_3x_4 \\
&\quad - 2B(G - 2b\gamma + \gamma\phi)x_2x_4 \\
&\quad - 2(G - 2b\gamma + \gamma\phi)x_1x_2 + 2Qx_2x_3 = 0,
\end{aligned} \tag{16}
$$

where the coefficients K, Q are given by

$$
\begin{aligned}
K &= b(G - 2b\gamma + \gamma\phi) - 3B(a - \theta_v) - B_v + q_v + \gamma p - bq \\
Q &= (2a - \theta_v)(G - 2b\gamma + \gamma\phi) + \tfrac{1}{2}(G - 2b\gamma + \gamma\phi)_v.
\end{aligned} \tag{17}
$$

Some properties of the quadrics $Q_1^{(u)}$, $Q_1^{(v)}$ will now be deduced. In the first place, the tangent plane, $x_4 = 0$, intersects each of the quadrics in a conic. The conic of intersection of the tangent plane and the quadric $Q_1^{(u)}$ touches the u-tangent at the point x and intersects the v-tangent in the point whose local coordinates are

$$
(\tfrac{1}{2}H, 0, F - 2a\beta + \beta\psi, 0). \tag{18}
$$

Similarly, the quadric $Q_1^{(v)}$ is intersected by the tangent plane in a conic which is tangent to the v-tangent at the point x and intersects the u-tangent in the point

(19) $$(\tfrac{1}{2}K, G - 2b\gamma + \gamma\phi, 0, 0).$$

The face $x_3=0$ of the tetrahedron of reference intersects the quadric $Q_1^{(u)}$ in the line l_1 and in the line whose equations are

(20) $$\beta x_2 + 2(F - 2a\beta + \beta\psi)x_4 = 0, \qquad x_3 = 0.$$

Moreover, the face $x_2=0$ cuts the quadric $Q_1^{(u)}$ in the line l_1 and in the line which joins the point (18) to the point on the line l_1 with local coordinates

(21) $$(-A, 0, 0, 1).$$

Similarly, the face $x_2=0$ cuts the quadric $Q_1^{(v)}$ in the line l_1 and in the line

(22) $$\gamma x_3 + 2(G - 2b\gamma + \gamma\phi)x_4 = 0, \qquad x_2 = 0.$$

The face $x_3=0$ cuts the quadric $Q_1^{(v)}$ in the line l_1 and in the line which passes through the point (19) and meets the line l_1 in the point

(23) $$(-B, 0, 0, 1).$$

The points (21), (23) are found to coincide if, and only if, $a_u=b_v$. Thus we reach the following conclusion:

The Γ_1-curves and the Γ_2-curves of two reciprocal rectilinear congruences form conjugate nets on the surface if, and only if, the points (21), (23) *coincide.*

It is well known that two nonsingular quadric surfaces having one, and only one, generator in common intersect elsewhere in a twisted cubic. Elimination of x_1 between equations (14), (16) gives the cubic cone projecting the curve of intersection of the two quadrics from the point x. This cone has the line l_1 for a double line, the equations of the nodal tangent planes along the line l_1 being given by

(24) $$(F - 2a\beta + \beta\psi)x_2^2 - (b_v - a_u)x_2x_3 - (G - 2b\gamma + \gamma\phi)x_3^2 = 0.$$

A glance at equation (5) suffices to substantiate the following statement:

The nodal tangent planes along the double line l_1 of the cone projecting the curve of intersection of the quadrics $Q_1^{(u)}$, $Q_1^{(v)}$ from the point x are the planes which intersect the tangent plane of the surface at the point x in the tangents of the Γ_1-curves.

Eliminating x_2 from equations (14), (16), we obtain the equation of the cone which projects the curve of intersection of the quadrics

$Q_1^{(u)}$, $Q_1^{(v)}$ from the vertex ρ of the tetrahedron of reference. This projecting cone is found to be a composite quartic cone, one component being the face $x_3=0$ of the tetrahedron of reference. The other component is a cubic cone which is intersected by the face $x_2=0$ in a plane cubic curve. Placing $x_3=0$ in the equation of this curve, we find the intersections of the curve with the line l_1. It is now easy to verify the conclusion:

The quadrics $Q_1^{(u)}$, $Q_1^{(v)}$ intersect in the line l_1 and in a twisted cubic which crosses the line l_1 in its two focal points.

4. **Osculating quadrics of ruled surfaces of the congruence Γ_2.** The equations of the quadrics $Q_2^{(u)}$, $Q_2^{(v)}$ can be found without difficulty by applying the method of the preceding section. The details of the calculation need not be reproduced here, but the required *equation of the quadric $Q_2^{(u)}$, referred to the tetrahedron x, ρ, σ, y,* is found to be

$$\beta x_1^2 - 2Fx_1x_3 + 2F^2x_3x_4 + 2(ab - a_u)Fx_3x_4 + 2Sx_1x_4 + Lx_4^2 = 0, \tag{25}$$

where the functions S, L are defined by

$$\begin{aligned} S &= \tfrac{1}{2}F_u - F\theta_u + 2bF - \beta(ab - a_u), \\ L &= \beta FG + FF_v - F(\theta_u - 2b)(b_v - a_u) + F(b_v - a_u)_u \\ &\quad - 2S(ab - a_u) - \beta(ab - a_u)^2. \end{aligned} \tag{26}$$

The equation of the quadric $Q_2^{(v)}$ is

$$\gamma x_1^2 - 2Gx_1x_2 + 2G^2x_3x_4 + 2(ab - b_v)Gx_2x_4 + 2Tx_1x_4 + Mx_4^2 = 0 \tag{27}$$

where

$$\begin{aligned} T &= \tfrac{1}{2}G_v - G\theta_v + 2aG - \gamma(ab - b_v), \\ M &= \gamma FG + GG_u - G(\theta_v - 2a)(a_u - b_v) + G(a_u - b_v)_v \\ &\quad - 2T(ab - b_v) - \gamma(ab - b_v)^2. \end{aligned} \tag{28}$$

The quadric $Q_2^{(u)}$ is intersected by the tangent plane in the line l_2 and also in the line

$$\beta x_1 - 2Fx_3 = 0, \qquad x_4 = 0. \tag{29}$$

The face $x_3=0$ cuts the quadric $Q_2^{(u)}$ in a conic which is tangent to the u-tangent at the point ρ and which intersects the edge $x_1=x_3=0$ in the point with local coordinates

$$(0, L, 0, -2F^2). \tag{30}$$

The face $x_1=0$ cuts the quadric $Q_2^{(u)}$ in the line l_2 and in the line which joins the point

$$(0, a_u - ab, F, 0) \tag{31}$$

on the line l_2 to the point (30).

Similarly, the tangent plane intersects the quadric $Q_2^{(v)}$ in the line l_2 and in the line

$$\gamma x_1 - 2Gx_2 = 0, \qquad x_4 = 0. \tag{32}$$

The plane $x_2=0$ cuts this quadric in a conic which is tangent to the v-tangent at the point σ and which intersects the edge $x_1=x_2=0$ in the point

$$(0, 0, M, -2G^2). \tag{33}$$

The face $x_1=0$ intersects the quadric $Q_2^{(v)}$ in the line l_2 and in the line which joins the point

$$(0, G, b_v - ab, 0) \tag{34}$$

on the line l_2 to the point (33). The following conclusion is immediate.

If the points (31), (34) *coincide respectively with the points* σ, ρ, *the* Γ_1*-curves and the* Γ_2*-curves form conjugate nets.*

Elimination of x_2 from equations (25), (27) yields the equation of the cubic cone projecting from the point ρ the curve of intersection of the quadrics $Q_2^{(u)}$, $Q_2^{(v)}$. The line l_2 is a double line of this cone, the nodal tangent planes along the line l_2 being given by

$$x_1^2 - (b_v - a_u)x_1x_4 - [FG + (ab - b_v)(ab - a_u)]x_4^2 = 0. \tag{35}$$

It is now easy to verify the conclusion:

The two nodal tangent planes along the double line l_2 *of the cone projecting the curve of intersection of the quadrics* $Q_2^{(u)}$, $Q_2^{(v)}$ *from the point* ρ *intersect the line* l_1 *in two points which separate the points* x, y *harmonically if, and only if, the* Γ_1*-curves and the* Γ_2*-curves form conjugate nets.*

Finally, simple calculations suffice to demonstrate the following theorem:

The quadrics $Q_2^{(u)}$, $Q_2^{(v)}$ *intersect in the line* l_2 *and in a twisted cubic which cuts the line* l_2 *in its two focal points.*

5. **A special case.** The theory of the preceding sections will now be specialized by considering a particular covariant pair of reciprocal

lines associated with the point x of the surface. It is known that the line l_1 is the projective normal and the line l_2 is the reciprocal projective normal in case $a=b=0$ in equations (3), (4). Placing $a=b=0$ in equations (14), (16), one easily shows that the equations of the two quadrics $Q_1^{(u)}$, $Q_1^{(v)}$, which we shall call *the projective normal quadrics*, are respectively

$$
\begin{aligned}
&\beta\pi x_2^2 + (p_v + \beta q + 3\theta_u k - k_u)x_2^2 + 2\pi^2 x_2x_4 + (\pi_u - 2\pi\theta_u)x_2x_3 \\
&\quad - 2\pi k x_3x_4 - 2\pi x_1x_3 = 0, \\
&\gamma\chi x_3^2 + (q_u + \gamma p + 3\theta_v k - k_v)x_2^2 + 2\chi^2 x_3x_4 + (\chi_v - 2\chi\theta_v)x_2x_3 \\
&\quad - 2\chi k x_2x_4 - 2\chi x_1x_2 = 0,
\end{aligned} \tag{36}
$$

where π, χ, k are defined by the formulas

$$
\pi = p + \beta\psi, \qquad \chi = q + \gamma\phi, \qquad k = \beta\gamma + \theta_{uv}.
$$

Moreover, by placing $a=b=0$ in equations (25), (27), we obtain *the two reciprocal projective normal quadrics* $Q_2^{(u)}$, $Q_2^{(v)}$, whose equations are respectively

$$
\begin{aligned}
&\beta x_1^2 + p(p_v + \beta q)x_4^2 + (p_u - 2p\theta_u)x_1x_4 - 2px_1x_3 + 2p^2 x_2x_4 = 0, \\
&\gamma x_1^2 + q(q_u + \gamma p)x_4^2 + (q_v - 2q\theta_v)x_1x_4 - 2qx_1x_2 + 2q^2 x_3x_4 = 0.
\end{aligned} \tag{37}
$$

SOUTHWESTERN COLLEGE

INDECOMPOSABLE CONNEXES[1]

PAUL M. SWINGLE

DEFINITION. *A connected set M is an indecomposable connexe if and only if, for every two connected subsets H and K of M such that $M=H+K$, either H and M or K and M have the same closure.*[2]

Any connected subset N of an indecomposable continuum W, which is dense in W, such as any set of composants of W or W itself, is an indecomposable connexe, as is also a widely connected set.[3]

EXAMPLE A.[4] Let, in a euclidean plane, U be the points of a square, Q, plus its interior. Let U_i $(i=1, 2, 3, \cdots)$ be a set of mutually exclusive arcs each contained in U and having one and only one point, an end point, common with Q. Let the U_i's be taken so that every plane region of U is joined to every linear region of Q by at least one U_i. Let $M=U-(U_1+U_2+\cdots)$. Then M is connected[5] and such that, if H and K are connected and their sum is M, either H and M or K and M have the same closure. Hence M is an indecomposable connexe.

EXAMPLE B. Let, in a euclidean plane, U be the points of a triangle plus its interior, one vertex of which is the point a. Let U_i $(i=1, 2, 3, \cdots)$ be a set of arcs, mutually exclusive, except for having the common end point a, and whose sum is dense in U. Let further the U_i's be taken so that each two plane regions of U are joined by at least one U_i. Let $M=U-(U_1+U_2+\cdots)$. It can be shown without difficulty that M is an indecomposable connexe.

---

[1] Presented to the Society November 23, 1940.

[2] See S. Eilenberg, *Topology du plan*, Fundamenta Mathematicae, vol. 26, p. 81, for a definition of an indecomposable connected space. This definition is seen to be equivalent to the above for the types of spaces considered in these two papers.

[3] For definition and example see P. M. Swingle, *Two types of connected sets*, this Bulletin, vol. 37 (1931), pp. 254–258.

[4] E. W. Miller communicated this interesting example to me by letter in 1937 calling attention to its relation to a widely connected set. The method of construction is somewhat similar to the well known boring process used to obtain a plane indecomposable continuum. See K. Yoneyama, *Theory of continuous sets of points*, Tôhoku Mathematical Journal, vol. 12 (1917), p. 60. That either H and M or K and M have the same closure is seen above by supposing that neither H nor K is dense in M, from which it readily follows that H and K can each have at most one point common with Q itself.

[5] E. W. Miller, *Some theorems on continua*, this Bulletin, vol. 46 (1940), p. 153, Theorem 3.

It is proposed to give here a generalization of some of the well known theorems on indecomposable continua[6] by means of indecomposable connexes and the following definitions. The imbedding space will be one satisfying R. L. Moore's Axioms 0 and 1.[7]

DEFINITIONS. *A connected subset K of a connected set M will be called a proper connexe subclosure of M if and only if M and K do not have the same closure. A connected set M is an irreducible connexe closure between two points a and b if and only if M contains $a+b$ and there does not exist a proper connexe subclosure of M containing $a+b$. A connected set M is an irreducible joining connexe closure between a and b if and only if there exists a subset N of M such that both N and $N+a+b$ are connected and, for all such N's, M and N have the same closure.*

Both a continuum and a connected set, irreducible between two points, are irreducible connexe closures between these two points. Also a widely connected set is an irreducible connexe closure between any two of its points. It is seen readily that *if M is an irreducible connexe closure* between a and b, then *M is an irreducible joining connexe closure between a and b.*

EXAMPLE C. In a euclidean plane let B be a biconnected set with dispersion point a and containing the point b distinct from a. Let W be an arc-wise connected set such that (a) if x and y are any two points of W then $W+a$ contains arcs ax and ay such that one of these contains the other, (b) for each x there exists but one arc ax, (c) the closure of $W+a-ax$ contains B, and (d) the product of ax and the closure of B is a. Then $M=W+B-a-b$ is an irreducible joining connexe closure from a to b, since each connected subset N of M, such that $N+a+b$ is connected, contains W. However $M+a+b$ is not an irreducible connexe closure from a to b, since $M+a+b$ contains B, which contains $a+b$, and B and M do not have the same closure.

DEFINITIONS. *A connected subset K of a connected set M is a connexe of condensation of M if and only if every point of K is a limit point of $M-K$. If M is connected a composant of $M+$ is a set of points K_p, which consists of a point p, of the closure of M but not necessarily of M, and of all points x of M such that there exists a proper connexe subclosure containing $p+x$ and contained in M excepting perhaps for p.*

[6] Brouwer, *Zur Analysis situs*, Mathematische Annalen, vol. 68 (1910), p. 426, gave the first example and definition of indecomposable continuum. For theorems on these sets see Z. Janiszewski and C. Kuratowski, *Sur les continus indécomposables*, Fundamenta Mathematicae, vol. 1, p. 215.

[7] *Foundations of Point Set Theory*, American Mathematical Society Colloquium Publications, vol. 13, 1932.

And K_p is a composant of M if and only if p is also contained in M, i.e., if K_p is a component of $M+$ but is contained entirely in M.

In a widely connected set M each composant of M consists of but one point, and each composant of $M+$ may consist of but one point. Hence it is not true that if K is such a composant of M every point of M is a limit point of K, which is however a useful theorem on indecomposable continua.[8]

THEOREM 107′. *Every composant of $M+$, where M is connected and its closure is compact, is the sum of a countable number of proper connexe subclosures, each contained in M except perhaps for one point of the closure of M.*

PROOF. Let a be a point of the closure of M and let K denote the composant of $M+$ consisting of a and all points x of M such that $M+a$ is not an irreducible connexe closure from a to x. Then there exists[9] a countable set G of domains such that if q is any point of the closure of M and D is any domain containing q there exists a domain of G, containing q, and contained wholly in D. For each domain R of G, which does not contain a, let M_R denote the maximal connected subset, containing a, of $(M+a)\cdot(S-R)$, S being the imbedding space. Let H denote the collection of all sets M_R and let T denote the sum of all these proper connexe subclosures of $M+a$ which are elements of H. The set H is countable. If q is a point of $M-T$ then $M+a$ is an irreducible connexe closure from a to q. For if there exists a proper connexe subclosure N of $M+a$, containing $a+q$, there exists a domain g of G such that the product of the closures of g and N is vacuous, where g contains a point of M. Thus N would have been contained in an M_R above and so N, and thus q, would be contained in T. Therefore K is T. Hence K is the sum of a countable number of proper connexe subclosures as the theorem states.

COROLLARY 107′. *If M is connected and its closure is compact, then every composant of M is the sum of a countable number of proper connexe subclosures of M.*

LEMMA A. *If M is an indecomposable connexe and N is a proper connexe subclosure of M, then $M-M\cdot\overline{N}$ is connected.*[10]

---

[8] R. L. Moore, loc. cit., Theorem 106, p. 75. Below, the theorems are numbered to correspond to similar theorems on indecomposable continua, given by Moore, pp. 75–78. It is to be noted the methods of proof are somewhat similar.

[9] R. L. Moore, loc. cit., Theorem 19, p. 14.

[10] By $\overline{N}$ is meant the closure of N.

PROOF. Suppose $M-M\cdot\overline{N}$ is the sum of the two mutually separated sets H and K. Then M is the sum of two proper connexe[11] subclosures $H+\overline{N}\cdot M$ and $K+\overline{N}\cdot M$ and so M is not indecomposable.

LEMMA A'. *If M is an indecomposable connexe and N is a proper connexe subclosure of M, then $M-N$ is connected.*

PROOF. By Lemma A $M-M\cdot\overline{N}$ is connected. Also $\overline{N}\cdot M$ is connected since N is. As M is the sum of these two sets and $M\cdot\overline{N}$ is a proper connexe subclosure, $M-M\cdot\overline{N}$ cannot be proper.

Suppose $M-N$ is the sum of the mutually separate sets U and V. But $M-N$ contains the connected set $M-M\cdot\overline{N}$ and so either U or V contains it also. Say U does. Then M and U must have the same closure. But then points of V are limit points of U which is a contradiction. Hence $M-N$ is connected.

THEOREM A. *If M is an indecomposable connexe and W a connected subset of M such that M and W have the same closure, then W is an indecomposable connexe.*

PROOF. Let $N=M-W$ and suppose $W=H+K$, H and K proper connexe subclosures of W. As N is contained in $\overline{W}=\overline{H}+\overline{K}$, let $\overline{H}\cdot N=H'$ and $\overline{K}\cdot N=K'$. Thus $H+H'$ and $K+K'$ are connected sets.[12] But as $\overline{H}$ contains the closure of $H+H'$ and $\overline{K}$ the closure of $K+K'$, $M=W+N$ is the sum of these two proper connexe subclosures and so M is not indecomposable.

COROLLARY A. *If M is an indecomposable connexe and N is both a proper connexe subclosure and a connexe of condensation of M, then $M-N$ is an indecomposable connexe.*

PROOF. By Lemma A $M-N$ is connected and by definition of connexe of condensation M and $M-N$ have the same closure. Thus the corollary follows from Theorem A.

COROLLARY A'. *If $M+f$ is an indecomposable connexe, M connected and f finite, then M is an indecomposable connexe.*

Theorem A and its corollaries treat the case where an indecomposable connexe is given and the *subtraction* of points gives an indecomposable connexe. This suggests the following *addition* problem: Let M be an indecomposable connexe and p a point of $\overline{M}-M$. *Is $M+p$ an indecomposable connexe?* This problem is left unsolved here.

---

[11] R. L. Moore, loc. cit., Theorem 47, p. 33.

[12] R. L. Moore, Theorem 27, p. 17.

THEOREM 108′. *Let M be connected. Then in order that M be an indecomposable connexe it is necessary and sufficient that every proper connexe subclosure of M be a connexe of condensation of M.*

PROOF. The condition is sufficient. For suppose M is not indecomposable. Then M is the sum of two proper connexe subclosures H and K. Thus there exists a point q of H which is not a limit point of K. But K contains $M-H$. Thus q is not a limit point of $M-H$ and so H is not a connexe of condensation of M.

The condition is necessary. For suppose N is a proper connexe subclosure of M but that not every point of N is a limit point of $M-N$. By Lemma A′ $M-N$ is connected but the closures of M and $M-N$ are not the same. Hence M is the sum of two proper connexe subclosures N and $M-N$ which is a contradiction.

THEOREM 108″. *Let M be connected. Then in order that M be an indecomposable connexe it is necessary and sufficient that the closure of every proper connexe subclosure of M be a continuum of condensation of the closure of M.*

PROOF. The condition is sufficient. For suppose H and K are as in the proof above and that q is a point of $\overline{H}$ which is not a limit point of K. As $\overline{M}=\overline{H}+\overline{K}$ and $q\cdot\overline{K}=0$ q is not a limit point of $\overline{M}-\overline{H}$ contained in $\overline{K}$. Thus $\overline{H}$ is not a continuum of condensation of $\overline{M}$.

The condition is necessary. As M is indecomposable, by Lemma A, $M-M\cdot\overline{N}$ is connected, where N is a proper connexe subclosure of M. Hence M is the sum of the two connected sets $M-M\cdot\overline{N}$ and $M\cdot\overline{N}$, the latter being a proper connexe subclosure of M. Hence $M-M\cdot\overline{N}$ is not proper and so every point of $M\cdot\overline{N}$, and so of N, is a limit point of $M-M\cdot\overline{N}$. Thus every point of $\overline{N}$ is a limit point of $M-M\cdot\overline{N}=(M+\overline{N})-\overline{N}$. Therefore every point of $\overline{N}$ is a limit point of $\overline{M}-\overline{N}$ and so $\overline{N}$ is a continuum of condensation of $\overline{M}$.

Let B be a composant of an indecomposable continuum K, where $K-B$ contains an arc A. Let c and d be two points of A such that $A-c-d=A'+A''+A'''$, where A', A'', and A''' are mutually separated sets, but $A'+c+A''$ and $A''+d+A'''$ are connected. Let $M=B+A'+A''+A'''$. Then M is an indecomposable connexe. The composant of $M+$ containing c is $A'+c+A''$ and the one containing d is $A''+d+A'''$. Thus two composants of $M+$ are not necessarily mutually exclusive.

THEOREM 109′. *If M is an indecomposable connexe, whose closure is compact, then no two composants of M have a point in common.*

PROOF. For each point p of M let M_p denote the set of points x such that M is not an irreducible connexe closure from p to x. If b is a point of M_a then $M_b = M_a$. For suppose not and that x is any point of M_a and y is of M_b. Then there exist proper connexe subclosures N_{ax}, N_{by}, N_{ab} of M. Suppose $\overline{N}_{ab} + \overline{N}_{ax} = \overline{M}$. But N_{ab} and N_{ax} are continua of condensation of $\overline{M}$ by Theorem 108''. This is a contradiction.[13] Therefore $N_{ab} + N_{ax}$ is a proper connexe subclosure of M as is similarly $(N_{ab} + N_{ax}) + N_{by}$. Hence $N_{ab} + N_{ax} + N_{by}$ is contained in both M_a and in M_b and so $M_a = M_b$. Hence if two composants have a point in common they are the same composant.

Since a composant of an indecomposable continuum is itself an indecomposable connexe it is not true that an indecomposable connexe contains uncountably many composants. A composant of $M+$ however may consist of a single point. Thus we have the following theorem.

THEOREM 110'. *If M is an indecomposable connexe whose closure is compact and, for every point p of $\overline{M} - M$, $M + p$ is an indecomposable connexe, then there exist an uncountable number of composants of $M+$.*

PROOF. Suppose there exist but a countable number of composants of $M+$. Then by Theorem 107' M is contained in a countable number of proper connexe subclosures of $\overline{M}$. Say these are the elements of the set (N). An N of (N) contains at most one point p of $\overline{M} - M$ and by hypothesis $M + p$ is an indecomposable connexe. Hence by Theorem 108'' $\overline{N}$ is a continuum of condensation of $\overline{M + p} = \overline{M}$. But $\overline{M}$ is the sum of the $\overline{N}$'s of (N), since M is the sum of the N's. As this is a contradiction[14] the theorem is true.

THEOREM 111'. *If M is connected and its closure is compact then in order that M be an indecomposable connexe it is necessary and sufficient that there exist three distinct points such that M is an irreducible joining connexe closure between any two of them.*

PROOF. The condition is sufficient. For if M is the sum of the connexes H and K, one of these has at least two of the three points as limit points and so it and M have the same closure.

The condition is necessary. For if M contains three points x, y, and z such that each of these is in a different composant, M is an irreducible connexe closure between any two of these points. Consider the case where M contains only the one composant T, containing a

[13] R. L. Moore, loc. cit., Theorem 15, p. 11.

[14] R. L. Moore, loc. cit., Theorem 15, p. 11.

point x. Then by Corollary 107′ M is the sum of the elements of a countable class (N), each element of which is a proper connexe subclosure. Then by Theorem 108″ every $\overline{N}$ of (N) is a continuum of condensation of $\overline{M}$. But if $\overline{M}$ is the sum of the $\overline{N}$'s this is a contradiction.[15] Hence $\overline{M}-M$ contains points y and z which are not contained in any $\overline{N}$ of (N). Thus if the connected set H of T contains x and has z as a limit point, H and M have the same closure. Thus M is an irreducible joining connexe closure from x to z and similarily from x to y. Suppose M contains a proper connexe subclosure N' which has y and z as limit points. Because of the nature of H above, N' does not contain x. From the manner of constructing the sets N of (N) in Theorem 107′, using x for the point a there, it is seen that N' is contained in an N of (N) and so does not have y or z as a limit point. Therefore M is an irreducible joining connexe closure between y and z also. In case M is the sum of two composants, y and z can be taken as above and the proof completed.[16]

THEOREM 112′. *If a is a point of an indecomposable connexe M whose closure is compact and K is the set of all points x such that M is an irreducible joining connexe closure from a to x, then K is dense in M.*

PROOF. Suppose that there exists a region R, containing a point of M, such that R does not contain a point of K. Let N be a maximal connected subset of $R\cdot M$. Then by Lemma A $M-M\cdot\overline{N}$ is connected as N is a proper connexe subclosure of M. Thus $\overline{N}$ is a continuum of condensation of $\overline{M}$. Hence[17] the locally compact closed set $\overline{M}\cdot\overline{R}$ is not the sum of the closures of a countable number of composants of $M\cdot R$. Hence by Theorem 107′ $\overline{M}\cdot\overline{R}$ is not contained in the sum of the closures of the countable number of proper connexe subclosures

[15] R. L. Moore, loc. cit., Theorem 15, p. 11.

[16] The question arises whether the condition in Theorem 111′ might be changed to "there exist three points x, y, and z such that $M+x+y+z$ is an irreducible connexe closure between any two of these." That three points might be taken so that $M+y$, say, is not an irreducible connexe closure between y and some point of M is seen by the following example. Let interior to the square Q, of Example A above, (V) be the set of straight line intervals joining a Cantor ternary set, on a line t, to a point y not on t. Take the U_i's as in Example A, except that no U_i has a point common with a V of (V). Let $B+y$ be a biconnected subset of the sum of the elements of (V), B being totally disconnected. Let $M=U-(U_1+U_2+\cdots)-(\text{points of the elements of } (V))+B$. Then M is indecomposable but $M+y$ is not an irreducible connexe closure between y and a point of B. See Example C above. Whether M could be taken so that each point of $\overline{M}-M$ is as y and $M+y$ is not an irreducible connexe closure between any two points is a question.

[17] R. L. Moore, loc. cit., Theorem 15, p. 11.

of the composant of M which contains a. Therefore by a proof similar to that of Theorem 111′ M is an irreducible joining connexe closure between a and some point of $(\overline{M-M})\cdot R$. Thus K is dense in M.

If T is the sum of a countable number of proper connexe subclosures of an indecomposable connexe M, since M may be a composant of an indecomposable continuum, it is readily seen that $M-T$ may be disconnected. However, by repeated use of Lemma A, Theorem 108′, and Theorem A, the following theorem is seen to be true.

THEOREM 113′. *If T is the sum of a finite number of mutually exclusive proper connexe subclosures of an indecomposable connexe M, then $M-T$ is a non-vacuous indecomposable connexe.*

The two following theorems are proven in a manner similar to that used for the corresponding theorems on continua.

THEOREM 114′. *If a is a point of a decomposable connexe M, there exists a domain D containing a such that M is not an irreducible connexe closure from a to any point of D.*

THEOREM 115′. *If a and b are two distinct points, M is an irreducible connexe closure from a to b, and T is a proper connexe subclosure of M containing b, then $M-M\cdot\overline{T}$ is connected.*

NEW MEXICO STATE COLLEGE

A PROBLEM IN PARTITIONS[1]

MARSHALL HALL

Let m objects $x_1, \cdots, x_m$ be given and from these n non-void subsets $a_1, \cdots, a_n$ be formed. This partition will determine a matrix (a_{ij}) in which $a_{ij}=1$ if a_i and a_j have a non-void intersection and $a_{ij}=0$ if a_i and a_j are disjoint. Necessarily (a_{ij}) is a symmetric matrix with 1's on the main diagonal. The following question has arisen in Ore's investigation of the theory of relations: Is every matrix (a_{ij}), $i, j=1, \cdots, n$, with $a_{ii}=1$, $a_{ij}=a_{ji}=0$ or 1 the partition matrix of n objects into n non-void subsets? As will be seen presently, the answer to this question is in the negative. The reason is not that there is any inherent contradiction within certain matrices but that it is not always possible to find a partition of as few as n objects determining a given matrix.

The answer is affirmative for $n=1, 2, 3, 4$ as may be found by direct calculation, but is negative for $n \geqq 5$. It is almost trivial that for $n \geqq 3$, $m=(n^2-n)/2$ objects will suffice. Take $(n^2-n)/2$ objects $u_{ij}=u_{ji}$, $i \neq j$, $i, j=1, \cdots, n$, and assign u_{ij} to both a_i and a_j if $a_{ij}=1$ and discard u_{ij} if $a_{ij}=0$. This will leave certain subsets a_i void for which $a_{ii}=1$, $a_{ij}=0$ if $j \neq i$, and for these we introduce new objects u_i in a_i alone. If there are one or two such i's we have discarded at least two u_{ij}'s since $n \geqq 3$. If there are $s \geqq 3$ such i's we have discarded at least $s(s-1)/2$ u_{ij}'s, namely those with both subscripts from this set. In all events we have discarded at least as many objects as we have added and we have a partition of $(n^2-n)/2$ or fewer objects into n non-void subsets corresponding to the prescribed partition matrix. But it is clear that this number $m=(n^2-n)/2$ is too high for $n>3$ since the full number of objects is used only if every $a_{ij}=1$ and in this case a single object assigned to every subset will suffice.

THEOREM 1. *A given matrix* (a_{ij}), $i, j=1, \cdots, n$, *in which* $a_{ii}=1$, $a_{ij}=a_{ji}=0$ *or* 1, *is the partition matrix of a set of at most n objects for* $n=1, 2, 3, 4$ *and of at most* $n^2/4$ (n *even and* $n \geqq 4$) *or* $(n^2-1)/4$ (n *odd and* $n \geqq 5$).

PROOF. Evidently renumbering the subsets makes no difference in the problem. This operation corresponds to permuting both the rows and columns of the matrix, the same permutation being applied to both. Such an operation defines an equivalence on the matrices. The

[1] Presented to the Society, February 22, 1941.

non-equivalent matrices of orders 2 and 3 are

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix},$$

$$(1) \qquad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}.$$

Those of order 2 require 2 and 1 objects respectively and those of order 3 require 3, 2, 2, and 1 objects respectively. There are eleven classes of matrices of order four of which only two,

$$(2) \qquad \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix},$$

require as many as four objects. Four classes require three, four classes require two, and one requires one.

Proof for $n=5$. If $a_{ij}=0$ for $j\neq i$, the matrix requires exactly five objects, one in each subset. If some $a_{ij}=1$ renumber to make this a_{12}, and suppress the first two rows and columns. The remaining matrix for a_3, a_4, and a_5 will require at most three objects, and only two unless $a_{34}=a_{35}=a_{45}=0$. Now take four objects u_{12}, u_3, u_4, u_5. Assign u_i, $i=3, 4, 5$, to a_1 and a_i if $a_{1i}=1$ and to a_2 and a_i if $a_{2i}=1$, and discard it if $a_{1i}=a_{2i}=0$. If there is an i for which $a_{1i}=a_{2i}=1$, then u_i is common to both a_1 and a_2 and we may discard u_{12}. If not assign u_{12} to both a_1 and a_2. These four objects and those used for the matrix for a_3, a_4, a_5 realize the partition with at most six elements save in the case in which exactly one of a_{1i}, a_{2i} is 1 for $i=3, 4, 5$ and $a_{34}=a_{35}=a_{45}=0$. Now we may discard the elements placed in a_3, a_4, a_5 originally and the partition is realized using only the four elements u_{12}, u_3, u_4, and u_5.

Proof for $n\geqq 6$. If $a_{ij}=0$ for $j\neq i$ we assign a single element x_i to each subset a_i and realize the partition with exactly n objects. If some $a_{ij}=1$, $j\neq i$, renumber so that this is a_{12}. Now suppress the first and second rows and columns. We have a partition matrix for $n-2$ subsets which, by an induction hypothesis, may be realized with at most $(n-2)^2/4$ or $[(n-2)^2-1]/4$ objects, depending on whether n is even or odd. We now use $n-1$ new objects $u_{12}, u_3, u_4, \cdots, u_n$. We assign u_{12} to both a_1 and a_2. For $i\geqq 3$ if $a_{1i}=1$, $a_{2i}=0$ we assign u_i to a_1

and a_i. If $a_{1i}=0$, $a_{2i}=1$ we assign u_i to a_2 and a_1. If $a_{1i}=a_{2i}=1$ we assign u_i to a_1, a_2, and a_i. If $a_{1i}=a_{2i}=0$ we discard u_i. This yields a realization of the prescribed partition, using at most $n-1$ more objects. Since $(n-2)^2/4+n-1=n^2/4$ and $[(n-2)^2-1]/4+n-1=(n^2-1)/4$, the truth of the theorem for $n-2$ implies the truth for n, and the truth of the theorem for $n=4$, 5 completes the proof by induction.

We note that the new objects used are never assigned to more than three subsets. For $n=5$ this is also true, and for $n=4$ an object may be assigned to all four subsets only if every a_{ij} is 1. But here four objects u_{123}, u_{124}, u_{134}, u_{234} assigned respectively to the three appropriate subsets realize the partition. Hence it is never necessary to assign an object to more than three subsets.

THEOREM 2. *If $n=2s$ is even, the symmetric matrix (a_{ij}) with $a_{ii}=1$; $a_{ij}=0$ for $j\neq i$ and i, $j=1,\cdots,s$; $a_{ij}=1$ for $i=1,\cdots,s$, and $j=s+1,\cdots,2s$; $a_{ij}=0$ for $j>i$ and i, $j=s+1,\cdots,2s$, is the partition matrix of $n^2/4$ but no fewer objects. If $n=2s+1$ is odd, the matrix (a_{ij}) with $a_{ii}=1$, $a_{ij}=0$ for $j>i$ and i, $j=1,\cdots,s$; $a_{ij}=1$ for $i=1,\cdots,s$ and $j=s+1,\cdots,2s+1$; $a_{ij}=0$ for $j>i$ and i, $j=s+1,\cdots,2s+1$, is the partition matrix of $(n^2-1)/4$ but no fewer objects. A partition matrix of order $n\geqq 5$ requiring $n^2/4$ objects (n even) or $(n^2-1)/4$ objects (n odd) is equivalent to the appropriate matrix above.*

PROOF. In any partition if an object x belongs to as many as three subsets a_i, a_j, a_k, then in the partition matrix we have $a_{ij}=a_{ik}=a_{jk}=1$. But in the above matrices $a_{ij}=a_{ik}=1$ (i, j, k different) always imply $a_{jk}=0$. For if $i\leqq s$ and $a_{ij}=a_{ik}=1$, then $j\geqq s+1$, $k\geqq s+1$ and $a_{jk}=0$; and if $i\geqq s+1$, $a_{ij}=a_{ik}=1$ then $j\leqq s$, $k\leqq s$ and $a_{jk}=0$. Hence the matrices above must come from partitions in which no object appears in more than two subsets. Thus in these matrices each 1 above the main diagonal corresponds to at least one object in common to two subsets and these objects must all be different. Hence these are the partition matrices of $n^2/4$ or $(n^2-1)/4$ objects but no fewer.

Consider a matrix (a_{ij}), $n\geqq 5$, requiring the maximum number of objects. If some $a_{ii}=1$ and all $a_{ij}=0$, $j\neq i$, then we may adjoin a single element for a_i to a partition for the remaining $n-1$ subsets and have a partition with less than the maximum number of objects. First suppose $n\geqq 6$. In the proof of Theorem 1 for $n\geqq 6$ note that if $a_{1i}=a_{2i}=0$ we discard u_i and have fewer than $n-1$ new objects to adjoin. Also if $a_{1i}=a_{2i}=1$ we may discard u_{12} since u_i is an object common to both a_1 and a_2. In these cases the partition requires fewer

than $n^2/4$ or $(n^2-1)/4$ objects. Since, by renumbering, any triple i, j, k may be taken into 1, 2, i, it follows that in a matrix requiring the maximum number of objects exactly two of a_{ij}, a_{ik}, a_{jk} must be 1's and the third must be 0 or all three must be 0. For a nonvanishing triple renumber so that the common subscript of the two 1's is 1 and so that the first row is of the form $a_{11}=1, a_{1i}=0, i=2, \cdots, t, a_{1i}=1, i=t+1, \cdots, n$. Here $t \leqq n-2$ since there are at least two 1's in this row. Now $a_{ij}=0$ if $j \neq i, i, j \geqq t+1$, since otherwise a_{1i}, a_{1j}, a_{ij} would all be 1's. Also $a_{ij}=1$ if $i \leqq t, j \geqq t+1$, since $a_{1i}=0, a_{1j}=1$. Finally $a_{ij}=0$ if $j \neq i$ and $i, j \leqq t$. For $a_{ni}=1, a_{nj}=1$ by the preceding argument. This completely determines the matrix which has a rectangle containing $t(n-t)$ 1's in the upper right-hand corner and also in the lower left-hand corner but 0's elsewhere apart from the main diagonal. By placing objects u_{ij} in a_i and a_j if $a_{ij}=1, j \neq i$, we may realize this partition using $t(n-t)$ objects. These objects are certainly different since none may be in three subsets. If $n=2s$ is even $t(n-t)<n^2/4$ unless $t=s$ which yields the matrix of the theorem. If $n=2s+1$, $t(n-t)<(n^2-1)/4$ unless $t=s$ or $s+1$. $t=s$ yields the matrix of the theorem, while $t=s+1$ yields an equivalent matrix. The permutation $(1, n)(2, n-1) \cdots (s, s+1)$ of the rows and columns interchanges these two matrices.

Finally suppose $n=5$. Here $(n^2-1)/4=6$. Suppose for some triple, say 1, 2, 3, $a_{12}=a_{13}=a_{23}=1$. Then as in the proof of Theorem 1 for $n=5$, to a partition of objects for a_3, a_4, a_5 we need add at most three objects. If these three subsets required only two objects, we have used at most five objects for the entire partition. If they required three objects, then $a_{34}=a_{35}=a_{45}=0$ and we may discard the object originally placed in a_3 and have a complete partition using at most five objects. Hence there is no triple i, j, k for which $a_{ij}=a_{ik}=a_{jk}=1$. Similarly if for some triple, say 1, 2, 3, $a_{12}=1, a_{13}=0, a_{23}=0$ we need to add at most three objects to the partition for a_3, a_4, a_5 and if this requires three objects then either someone of the old objects may be discarded or one of the new objects is discarded. Hence if $a_{12}=1$, one of a_{13}, a_{23} is 1 and the other 0. From here the proof as for $n \geqq 6$ applies.

YALE UNIVERSITY

MEAN-VALUE SURFACES

MAXWELL READE AND E. F. BECKENBACH

Introduction. The real functions

$$x_j = x_j(u, v), \qquad j = 1, 2, 3, \tag{1}$$

defined and continuous in a finite simply connected domain[1] D, will be said to define a surface S. If the first partial derivatives of the functions (1) are continuous in D, and if

$$E(u, v) = G(u, v), \qquad F(u, v) = 0 \tag{2}$$

hold in D, where

$$E(u, v) \equiv \sum_{j=1}^{3} x_{ju}^2, \qquad F(u, v) \equiv \sum_{j=1}^{3} x_{ju}x_{jv}, \qquad G(u, v) \equiv \sum_{j=1}^{3} x_{jv}^2$$

are the coefficients of the first fundamental quadratic form of S, then the surface is said to be given in isothermic representation by the functions (1) and the parameters u, v are said to be isothermic parameters; the map of D on S is conformal except where $E=G=0$.

In a previous paper,[2] the authors studied the equation

$$\sum_{j=1}^{3}\left[\int_C x_j(u, v)dz\right]^2 = 0, \qquad z = u + iv, \tag{3}$$

where C is a circle in D; the following necessary and sufficient condition was obtained.

THEOREM A. *If the functions* (1) *have continuous partial derivatives of the third order in a finite simply connected domain D, then a necessary and sufficient condition that they map D isothermically either on a surface S that lies on a sphere of finite non-null radius, such that circles are mapped on circles, or on a minimal surface S, is that* (3) *hold for each circle C in D.*

1. **Mean-value surfaces.** Let the coordinate functions (1) of a surface S be continuous in a finite simply connected domain D; then the circular averages

[1] A domain is a non-null connected open set.

[2] *Generalizations to space of the Cauchy and Morera theorems*, Transactions of this Society, vol. 49 (1941), pp. 354–377; in particular, see p. 365.

$$(4)\quad x_j = A_{j,\rho}(u, v) \equiv \frac{1}{\pi\rho^2}\iint_{\xi^2+\eta^2\leq\rho^2} x_j(u+\xi, v+\eta)\,d\xi\,d\eta, \quad j = 1, 2, 3,$$

where ρ is a positive constant, will be said to define *a mean-value surface S_ρ associated with S*. We define

$$A_{j,0}(u, v) \equiv x_j(u, v), \qquad j = 1, 2, 3.$$

We note that the functions (4) are defined and have continuous partial derivatives of the first order in an open set of points D_ρ which is interior to D; since D_ρ is not necessarily a connected set, S_ρ may consist of several pieces.

THEOREM 1.1. *If the functions* (1) *are continuous in a simply connected domain D, then a necessary and sufficient condition that* (3) *hold for each circle C in D is that all mean-value surfaces S_ρ associated with the surface S, defined by the functions* (1), *be given in isothermic representation by* (4).

PROOF. The first partial derivatives of the functions (4) are given by the relations[3]

$$\frac{\partial A_{j,\rho}}{\partial u} = \frac{1}{\pi\rho^2}\int_{\xi^2+\eta^2=\rho^2} x_j(u+\xi, v+\eta)\,d\eta,$$

$$\frac{\partial A_{j,\rho}}{\partial v} = -\frac{1}{\pi\rho^2}\int_{\xi^2+\eta^2=\rho^2} x_j(u+\xi, v+\eta)\,d\xi, \quad j = 1, 2, 3,$$

which are valid for points of D_ρ; hence

$$(5)\quad \sum_{j=1}^{3}\left[\int_{\xi^2+\eta^2=\rho^2} x_j(u+\xi, v+\eta)(d\xi + i\,d\eta)\right]^2 = -\pi^2\rho^4[E_\rho - G_\rho + 2iF_\rho],$$

where E_ρ, F_ρ and G_ρ are the coefficients of the first fundamental quadratic form of S_ρ. From (2), (3) and (5) we obtain the theorem.

From Theorems A and 1.1 we obtain the following result.

THEOREM 1.2. *If the functions* (1) *have continuous partial derivatives of the third order in a finite simply connected domain D, then a necessary and sufficient condition that they map D isothermically either on a surface S that lies on a sphere of finite non-null radius, such that circles are mapped on circles, or on a minimal surface S, is that all mean-value surfaces S_ρ associated with the surface S defined by the functions* (1) *be given in isothermic representation by* (4).

[3] T. Radó, *Subharmonic Functions*, Berlin, 1937, p. 11.

2. **Mean-value surfaces and transformations of axes.** In §3 we shall make use of the following observations.

2.1. S_ρ is invariant under rigid transformations in the (x_1, x_2, x_3)-space; if

$$x_j = a_j + \sum_{k=1}^{3} \lambda_{kj} x_k, \qquad j = 1, 2, 3,$$

is a rigid transformation, then

$$A'_{j,\rho}(u, v) = a_j + \sum_{k=1}^{3} \lambda_{kj} A_{k,\rho}(u, v), \qquad j = 1, 2, 3,$$

where

$$A'_{j,\rho}(u, v) \equiv \frac{1}{\pi\rho^2} \iint_{\xi^2+\eta^2 \le \rho^2} x'_j(u + \xi, v + \eta) d\xi d\eta, \qquad j = 1, 2, 3.$$

2.2. S_ρ is invariant under each of the reflections

$$u' = u, \; v' = -v,$$

and

$$u' = -u, \; v' = v.$$

If, for example,

$$x'_j(u', v') \equiv x_j(u', -v'), \qquad j = 1, 2, 3,$$

then

$$A'_{j,\rho}(u', v') = A_{j,\rho}(u', -v'), \qquad j = 1, 2, 3,$$

where

$$(6) \quad A'_{j,\rho}(u', v') \equiv \frac{1}{\pi\rho^2} \iint_{\xi^2+\eta^2 \le \rho^2} x'_j(u' + \xi, v' + \eta) d\xi d\eta, \quad j = 1, 2, 3.$$

2.3. Similarly, S_ρ is invariant under rigid transformations in the (u, v)-plane.

2.4. Under the transformation

$$(7) \qquad u' = \alpha u, \quad v' = \alpha v, \qquad \alpha > 0,$$

the mean-value surface S_ρ is transformed into the mean-value surface $S'_{\alpha\rho}$ whose coordinate functions are given by (6), where

$$x'_j(u', v') \equiv x_j\left(\frac{u'}{\alpha}, \frac{v'}{\alpha}\right), \qquad j = 1, 2, 3.$$

Hence for α fixed, the *family* of mean-value surfaces $[S_\rho]$ associated with a given surface S is identical with the *family* of mean-value surfaces whose coordinate functions are given by (6); under the transformation (7), each member of one family is congruent to a member of the second family.

3. **Conformal mean-value surfaces.** Since, by Theorem 1.2, the only smooth surfaces in isothermic representation for which all associated mean-value surfaces are given in isothermic representation by (4) are (a) spherical surfaces, in representation whereby circles are mapped on circles, and (b) minimal surfaces, the question arises as to the nature of the mean-value surfaces in these two cases.

THEOREM 3.1. *If the functions* (1) *map a finite simply connected domain D isothermically on a minimal surface S, then each mean-value surface S_ρ associated with S is a minimal surface given in isothermic representation by* (4) *and coinciding with S for (u, v) in D_ρ.*

PROOF. By a theorem of Weierstrass,[4] the functions (1) are harmonic in D; consequently it follows from the mean-value property of harmonic functions that the functions (4) coincide with the functions (1) in the open set D_ρ. Hence all mean-value surfaces associated with minimal surfaces given in isothermic representation by (1) are themselves minimal surfaces given in isothermic representation by (4); the surface S_ρ coincides with S for (u, v) in D_ρ.

THEOREM 3.2. *If the functions* (1) *are not identically constant and if they map a finite simply connected domain D isothermically on a surface S that lies on a sphere $\mathfrak{S}$ of finite non-null radius a, such that circles are mapped on circles, then each mean-value surface S_ρ associated with S lies on a surface of revolution T_ρ and is given in isothermic representation by* (4). *Further, for $0<\rho<\infty$, T_ρ is not a sphere.*

PROOF. It has been pointed out, in a recently published paper,[5] that the functions (1) may be continued isothermically to map the entire closed u, v-plane isothermically on the whole of $\mathfrak{S}$; further, the functions (1) have the representation

[4] If the functions (1) are harmonic in D, and if (2) holds in D, then the functions (1) are said to form *a triple of conjugate harmonic functions*; see E. F. Beckenbach and T. Radó, *Subharmonic functions and minimal surfaces*, Transactions of this Society, vol. 35 (1933), pp. 648–661. Then the theorem of Weierstrass may be stated as follows. *A necessary and sufficient condition that the functions* (1), *defined in the domain D, be the coordinate functions of a minimal surface given in isothermic representation is that they form a triple of conjugate harmonic functions*; loc. cit., p. 649.

[5] Loc. cit., see Footnote 2; see p. 375.

$$
\begin{aligned}
x_1 &= x_1(u, v) \equiv a_1 + \frac{2a\,|k|\,\Re f(z)}{|f(z)|^2 + k^2}, \\
x_2 &= x_2(u, v) \equiv a_2 + \frac{2a\,|k|\,\Im f(z)}{|f(z)|^2 + k^2}, \\
x_3 &= x_3(u, v) \equiv k\left[1 - \frac{2a\,|k|}{|f(z)|^2 + k^2}\right],
\end{aligned} \tag{8}
$$

where $f(z)$ has one of the following forms:

$$f(z) \equiv \alpha z + \beta, \qquad f(z) \equiv \alpha \bar{z} + \beta,$$

where α and β are constants. In (8), $k = a_3 \pm a$, $|k|$ is the maximum of the two quantities $|a_3 + a|$ and $|a_3 - a|$ and (a_1, a_2, a_3) are the coordinates of the center of $\mathfrak{S}$.

From 2.1–2.3, it follows that we may assume $a_1 = a_2 = a_3 = 0$, $k = a$, in (8), and that we may assume the function $f(z)$ is given by

$$f(z) \equiv \alpha z, \qquad \alpha > 0. \tag{9}$$

This is equivalent to assuming that the point $z = \infty$ corresponds to the point P: (0, 0, a) on $\mathfrak{S}$, that the point $z = 0$ corresponds to the point P': (0, 0, $-a$) which is diametrically opposite P, and that the point in the z-plane corresponding to (a, 0, 0) is real and positive. Since we are investigating *all* mean-value surfaces associated with S, it follows from 2.4 that we may take $\alpha = 1$ in (9), in which case the functions (1) have the following familiar representation, as given by (8):

$$
\begin{aligned}
x_1 &= x_1(u, v) \equiv \frac{2a^2 u}{u^2 + v^2 + a^2}, \\
x_2 &= x_2(u, v) \equiv \frac{2a^2 v}{u^2 + v^2 + a^2}, \\
x_3 &= x_3(u, v) \equiv a\left[1 - \frac{2}{u^2 + v^2 + a^2}\right].
\end{aligned} \tag{10}
$$

If C_r is the circle $u^2 + v^2 = r^2$, and if

$$z = u + iv = re^{i\theta},$$

then (4) and (10) yield

$$
\begin{aligned}
A_{1,\rho}(u, v) + iA_{2,\rho}(u, v) &= e^{i\theta}[A_{1,\rho}(r, 0) + iA_{2,\rho}(r, 0)], \\
A_{3,\rho}(u, v) &= A_{3,\rho}(r, 0),
\end{aligned}
$$

from which it follows that the map of C_r on the mean-value surface T_ρ, associated with the sphere $\mathfrak{S}$ which is defined by (10), is a circle C_r^* in a plane perpendicular to the x_3-axis; moreover, the center of C_r^* is on the x_3-axis. Since S_ρ lies on T_ρ, it follows that the functions (4) map D_ρ isothermically on a surface that lies on the surface of revolution T_ρ.

In §4 it will appear that for $0<\rho<\infty$, T_ρ is not a sphere. Nevertheless, we shall call T_ρ *a mean-value sphere.*

4. **Mean-value spheres.** The mean-value sphere T_ρ is a surface of revolution about the x_3-axis. Accordingly, to investigate T_ρ, it is sufficient to study the intersection T_ρ^* of T_ρ with the plane $x_2=0$. Since, by (10),

$$x_2(u, -v) \equiv -x_2(u, v),$$

it follows that the intersection of T_ρ with the plane $x_2=0$ can be obtained from (4) by setting $v=0$ in (10). A computation yields the following coordinate functions for T_ρ^*:

$$\begin{aligned} x_1 &= A_{1,\rho}(u, 0) \\ &\equiv 4a^2u/\left[a^2+\rho^2+u^2+((a^2+\rho^2-u^2)^2+4a^2u^2)^{1/2}\right], \\ x_3 &= A_{3,\rho}(u, 0) \\ &\equiv a-\frac{2a^2}{\rho^2}\log\frac{a^2+\rho^2-u^2+((a^2+\rho^2-u^2)^2+4a^2u^2)^{1/2}}{2a^2}. \end{aligned} \tag{11}$$

We make the following observations.

4.1. The curve T^* is symmetric about the x_3-axis; since

$$A_{1,\rho}(u, 0) \equiv A_{1,\rho}\left(\frac{u^2+\rho^2}{u}, 0\right),$$

$$A_{3,\rho}(u, 0) \equiv 2a-\frac{2a^2}{\rho^2}\log\frac{a^2+\rho^2}{a^2}-A_{3,\rho}\left(\frac{u^2+\rho^2}{u}, 0\right),$$

it follows that T_ρ^* is also symmetric about the line

$$x_3 = A_{3,\rho}((a^2+\rho^2)^{1/2}, 0) = a-\frac{a^2}{\rho^2}\log\frac{a^2+\rho^2}{a^2}. \tag{12}$$

From this symmetry and from the relations,

$$m=\frac{2au}{a^2+\rho^2-u^2}, \qquad \frac{dm}{du}=\frac{2a(a^2+\rho^2+u^2)}{(a^2+\rho^2-u^2)^2},$$

satisfied by the slope m, it follows that T_ρ^* is convex.

4.2. The height h and the width l of T_ρ^* are given by

$$h = \frac{2a^2}{\rho^2} \log \frac{a^2 + \rho^2}{a^2}, \qquad l = \frac{4a^2}{(a^2 + \rho^2)^{1/2} + a};$$

hence *for* $0 < \rho < \infty$ *the curve* T_ρ^* *is not a circle.* Moreover, since

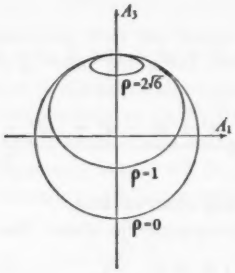
$$\lim_{\rho \to \infty} \frac{l}{h} = \infty,$$

it follows that T_ρ^* "flattens out" while approaching the point $(0, a)$, as $\rho \to \infty$.

4.3. Each member of the family $[T_\rho^*]$ passes through the point $(0, a)$ and is tangent there to every other member of the family. For $\rho = 0$, T_ρ^* is the circle $x_1^2 + x_3^2 = a^2$, and for $\rho = \infty$, T_ρ^* is the point $(0, a)$.

4.4. A computation shows that for $0 \leqq \rho < \rho'$, $T_{\rho'}^*$ is inside T_ρ^*, except for their common point of tangency.

The figure shows T_ρ^* for $\rho = 0,\ 1,\ 2(6)^{1/2},\ \infty\ ;\ a = 1$.



From 4.1–4.4 it follows that the family of surfaces $[T_\rho]$ consists of convex surfaces each of which passes through the point $(0, 0, a)$ in (x_1, x_2, x_3)-space and is tangent there to every other member of the family. The surface T_ρ is a surface of revolution about the x_3-axis and is symmetric with respect to the plane

$$x_3 = a - \frac{a^2}{\rho^2} \log \frac{a^2 + \rho^2}{a^2}.$$

Since the ratio of width to height $\to \infty$ as $\rho \to \infty$, it follows that T_ρ "flattens out" as $\rho \to \infty$. For $\rho = 0$, T_ρ is the sphere about the origin with radius a, and for $\rho = \infty$, T_ρ is the point $(0, 0, a)$. For $0 \leqq \rho' < \rho$, T_ρ is inside $T_{\rho'}$, except for their common point of tangency. From (11) we obtain the following isothermic representation for T_ρ:

$$x_1 = A_{1,\rho}(u, v)$$
$$\equiv \frac{a^2 u}{\rho^2(u^2+v^2)}\left[a^2+u^2+v^2+\rho^2-((a^2+\rho^2-u^2-v^2)^2+4a^2(u^2+v^2))^{1/2}\right],$$
$$x_2 = A_{2,\rho}(u, v)$$
$$\equiv \frac{a^2 v}{\rho^2(u^2+v^2)}\left[a^2+u^2+v^2+\rho^2-((a^2+\rho^2-u^2-v^2)^2+4a^2(u^2+v^2))^{1/2}\right],$$
$$x_3 = A_{3,\rho}(u, v)$$
$$\equiv a-\frac{2a^3}{\rho^2}\log\left[\frac{a^2-u^2-v^2+\rho^2+((a^2+\rho^2-u^2-v^2)^2+4a^2(u^2+v^2))^{1/2}}{2a^2}\right].$$

THE OHIO STATE UNIVERSITY AND
THE UNIVERSITY OF MICHIGAN

NOTE ON THE DISTRIBUTION OF VALUES OF THE ARITHMETIC FUNCTION $d(m)$[1]

M. KAC

1. **Introduction.** Recently Dr. Erdös and the present writer[2] proved the following theorem:

If $\nu(m)$ denotes the number of different prime divisors of m and $k_n(\omega)$ the number of positive integers $m \leqq n$ for which

$$\nu(m) \leqq \lg\lg n + \omega(2\lg\lg n)^{1/2},$$

then

$$\lim_{n\to\infty}\frac{k_n(\omega)}{n} = \pi^{-1/2}\int_{-\infty}^{\omega} e^{-u^2}du = D(\omega).$$

The purpose of this note is to derive a similar theorem concerning the function $d(m)$ which denotes the number of all different divisors of m (1 and m are included).

In fact we are going to prove the following theorem:

If $r_n(\omega)$ denotes the number of positive integers $m \leqq n$ for which

$$d(m) \leqq 2^{\lg\lg n+\omega(2\lg\lg n)^{1/2}},$$

---

[1] Presented to the Society, May 2, 1941.

[2] P. Erdös and M. Kac, *The Gaussian law of errors in the theory of additive number theoretic functions*, American Journal of Mathematics, vol. 62, pp. 738–742.

then

$$\lim_{n\to\infty} \frac{r_n(\omega)}{n} = \pi^{-1/2} \int_{-\infty}^{\omega} e^{-u^2} du = D(\omega).$$

2. **Proof of the theorem.** The proof is based on the theorem cited in the introduction and on the following two facts:

I. The mean value

$$M\{d(m)/2^{\nu(m)}\} = \lim_{n\to\infty} n^{-1} \sum_{m=1}^{n} d(m)/2^{\nu(m)}$$

exists and is finite.

II. If $f(m) \geqq 0$ is such that $M\{f(m)\}$ is finite, if $\lim g(n) = \infty$ as $n \to \infty$ and if $p(n)$ denotes the number of positive integers $m \leqq n$ for which $f(m) \leqq g(n)$, then $\lim p(n)/n = 1$ as $n \to \infty$.

I is implied by a theorem of E. R. van Kampen and A. Wintner[3] and II is almost obvious even under a weaker condition that $\limsup n^{-1} \sum_1^n f(m) < \infty$.

Let ω be an arbitrary real number and $\epsilon > 0$. Put $f(m) = d(m)/2^{\nu(m)}$ and $g(n) = 2^{\epsilon(2 \lg \lg n)^{1/2}}$. Let F_n be the set of positive integers $m \leqq n$ for which $\nu(m) \leqq \lg \lg n + (\omega - \epsilon)(2 \lg \lg n)^{1/2}$, G_n the set of positive integers $m \leqq n$ for which $f(m) \leqq g(n)$ and H_n the set of positive integers $m \leqq n$ for which $d(m) \leqq 2^{\lg \lg n + \omega(2 \lg \lg n)^{1/2}}$. If $m \in F_n G_n$ then $m \in H_n$. Hence,

$$F_n G_n \subset H_n.$$

The number of elements in F_n is $k_n(\omega - \epsilon)$; in G_n, $p(n)$; and in H_n, $r_n(\omega)$.

Thus, the number of elements in $F_n G_n$ is $\geqq k_n(\omega - \epsilon) - (n - p(n))$ and finally

$$k_n(\omega - \epsilon) - (n - p(n)) \leqq r_n(\omega).$$

On the other hand for every m, $2^{\nu(m)} \leqq d(m)$ (the equality occurs only if m is a prime) and therefore $H_n \subset F_n$ or $r_n(\omega) \leqq k_n(\omega)$. The inequalities combined give

$$k_n(\omega - \epsilon) - (n - p(n)) \leqq r_n(\omega) \leqq k_n(\omega).$$

But as $n \to \infty$ $k_n(\omega - \epsilon)/n \to D(\omega - \epsilon)$, $k_n(\omega)/n \to D(\omega)$ and $(n - p(n))/n \to 0$ (see I and II); hence

$$D(\omega - \epsilon) \leqq \liminf_{n\to\infty} \frac{r_n(\omega)}{n} \leqq \limsup_{n\to\infty} \frac{r_n(\omega)}{n} \leqq D(\omega).$$

[3] American Journal of Mathematics, vol. 62, p. 618 (Theorem IV).

Since ϵ is arbitrary and $D(\omega)$ is a continuous function of ω,

$$\lim_{n\to\infty} \frac{r_n(\omega)}{n} = D(\omega).$$

3. **Some numerical results.** The values of $d(m)$ for $1 \leqq m \leqq 10^4$ can be found in the recent tables of the British Association for the Advancement of Science,[4] so that it was easy to obtain the exact value of $10^{-4}r_{10^4}(\omega)$ for different values of ω and compare them with the values of $D(\omega)$ computed from the tables of the probability integral.[5]

$2^{\lg\lg n+\omega(2\lg\lg n)^{1/2}}$	ω	$10^{-4}r_{10^4}(\omega)$	$D(\omega)$
1	−1.054	0.0001	0.0680
2	−0.579	0.1230	0.3065
3	−0.302	0.1255	0.4346
4	−0.105	0.3863	0.4410
5	0.048	0.3867	0.5220
6	0.173	0.4631	0.5961
7	0.279	0.4633	0.6534
8	0.370	0.6747	0.6996
9	0.451	0.6779	0.7382
10	0.523	0.6929	0.7702
11	0.588	0.7970	0.7971
12	0.648	0.7971	0.8202
13	0.702	0.8012	0.8396
14	0.753	0.8027	0.8565
15	0.800	0.8827	0.8710
16	0.845	0.8827	0.8840

For a good agreement $n = 10{,}000$ seems to be too small. The rather striking fact that $10^{-4}r_{10^4}(.588)$ is almost equal to $D(.588)$ is probably accidental. The case $\omega = .800$ disproves the conjecture that always $n^{-1}r_n(\omega) \leqq D(\omega)$.

CORNELL UNIVERSITY

[4] *Mathematical Tables*, vol. 8, *Number-Divisor Tables*, Cambridge University Press, 1940. We refer in particular to Table III. In these tables $d(m)$ is denoted by $\nu(m)$.

[5] We used the tables on pages 388 to 391 of the first volume of Czuber's *Wahrscheinlichkeitsrechnung*, Teubner, 1908. It should be noted that $D(\omega) = \{1+\Phi(\omega)\}/2$.

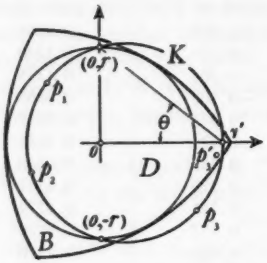
We also wish to thank Mr. W. J. Harrington for his help in computing the table given on this page.

A CHARACTERIZATION OF THE DISC

C. V. ROBINSON[1]

In this paper the disc[2] is characterized as the *only* connected, simply connected domain[3] B with the following property, C_3: B will cover (by an isometry) any subset P of the plane provided B will cover each 3 points of P.

The disc has property C_3. For a plane set P can be covered by a ρ-disc if and only if the members of the family F of ρ-discs with centers in P have a common point. If now each three points of P are



on a ρ-disc then each three discs of F intersect and by a theorem on convex bodies due to E. Helly[4] there is a point common to all the discs of F.

LEMMA. *A bounded, closed subset of the plane contains a largest circle.*

The proof is accomplished by selecting a sequence of circles from the set whose centers converge to a point and whose radii converge to the least upper bound of the radii of circles in the set and (using the fact that the set is closed) showing that the limiting circle belongs to the set.

THEOREM. *The disc is the only connected, simply connected domain with property* C_3.

PROOF. We assume B to be the given domain, and show that B

[1] Part of a Ph.D. dissertation at University of Missouri, under L. M. Blumenthal, 1940.

[2] The disc of radius ρ and center p is the set of points x of E_2 such that $px \leqq \rho$.

[3] Closure of a bounded open subset of E_2.

[4] *Theorem.* If each $n+1$ sets of a family of *bounded*, closed, convex subsets of E_n intersect, there is a point common to all the sets. Jahrbuch der Deutschen Mathematiker-Vereinigung, vol. 32 (1932), pp. 175–176.

must be a disc. By the preceding lemma, B contains a largest circle (center o and radius r), and since B is simply connected, the whole disc D of the circle is contained in B. If we suppose B is not a disc, there is a point p of B outside D, and, since B is connected, a point v' of B such that $r < ov' \leqq \max(op, 3^{1/2}r)$. (The inequality on the right insures that $\theta \geqq 30°$ in the figure.) Introduce coordinates in the plane with o as origin and the ray ov' as positive X-axis. Then D is defined by the inequality

$$D: \quad x^2 + y^2 \leqq r^2, \tag{1}$$

and the circle K of the three points $(0, \pm r)$, v' has the equation

$$K: \quad \left(x - \left(\frac{v^2 - r^2}{2v}\right)\right)^2 + y^2 = \left(\frac{v^2 + r^2}{2v}\right)^2 \tag{2}$$

where v is the abscissa of the point v'.

The circle K has the following properties:

(α) The radius of K is greater than r. For, since $v > r$,

$$\left(\frac{v^2 + r^2}{2v}\right) = \left(\frac{v^2 - r^2}{2v}\right)^2 + r^2 > r^2.$$

(β) The arc of K to the left of the Y-axis is of length $\geqq 120°$. For the arc is $2(2\theta) = 4\theta \geqq 120°$ by choice of v'.

(γ) The arc of K to the left of the Y-axis is in D. From (2)

$$x^2 + y^2 = r^2 + x\left(\frac{v^2 - r^2}{v}\right) \tag{3}$$

so that for $x \leqq 0$ the inequality (1) is satisfied by points of K.

(δ) The maximum distance of a point of K from the origin is v. This follows easily from (3).

If, now, any three points p_1, p_2, p_3 of K are selected, at least two of them, say p_1, p_2, will lie on an arc of length $\leqq 120°$. By property (β), K can then be rotated about its center so that this arc will fall to the left of the Y-axis, that is, $p_1, p_2 \in D$ by property (γ). Since $op_3 \leqq v$ by property (δ), there is, since B is connected, a point p_3' of B at distance op_3 from o. Rotate K about the origin so that p_3 goes into p_3'. Since p_1, p_2 will remain in D, the three points p_1, p_2, p_3 are seen to be congruent to three points of B. But p_1, p_2, p_3 are *any* three points of K, so that B must, by our assumption, cover K. By property (α) this is impossible, and hence no point p of B can lie outside D, that is, B is the disc D.

UNIVERSITY OF MISSOURI

ON SOME PROPERTIES OF SYMMETRICAL PERFECT SETS[1]

R. SALEM

This paper deals with some properties of symmetrical perfect sets, in view of applications: (i) to the problem of multiplicity of trigonometrical series; (ii) to the study of Fourier-Stieltjes coefficients of singular monotone functions; (iii) to the problem of absolute convergence of trigonometrical series.

1. **Sets of multiplicity of trigonometrical series.** We shall consider, throughout the paper, symmetrical perfect sets, that is, sets which are obtained, in the closed interval $(0, 2\pi)$ by the following process. We divide the interval in three parts of lengths proportional to ξ_1, $1-2\xi_1$, ξ_1, and we remove the central open interval. In the second operation we divide each one of the two intervals left in three parts proportional to ξ_2, $1-2\xi_2$, ξ_2, and we remove the two central open intervals, and so on infinitely, the sequence $\{\xi_p\}$ being such that $0<\xi_p\leqq\frac{1}{2}$.[2] After p operations, we have thus removed 2^p-1 intervals which we shall denote by δ_{pk} $(k=1, 2, \cdots, 2^p-1)$ and 2^p intervals are left, which we shall denote by η_{pk} $(k=1, 2, \cdots, 2^p)$. Each interval η_{pk} is of length equal to $\eta_p=2\pi\cdot\xi_1\xi_2\cdots\xi_p$. Let E_p be the set constituted by the 2^p intervals η_{pk} and let $E(p)$ be its measure. We have

$$E(p) = 2^p\eta_p = 2\pi\cdot 2^p\xi_1\xi_2\cdots\xi_p. \tag{1}$$

$E(p)$ is a non-increasing function of p and the measure of the perfect set P obtained by the above-described process is $\lim E(p)$ for $p=\infty$. We shall only consider, throughout the paper, sets for which this limit is equal to zero.

We shall now construct a monotone continuous function $F(x)$ constant in every interval contiguous to P but increasing from one interval to another, by the following well known process.[3] For every p let F_p be a non-decreasing continuous function defined by the following conditions: $F_p(0)=0$, $F_p(2\pi)=1$,

$$F_p(x) = \frac{k}{2^p} \text{ in } \delta_{pk}, \qquad k = 1, 2, \cdots, 2^p-1,$$

---

[1] Presented to the Society, May 3, 1941.

[2] If $\xi_p=\frac{1}{2}$ no interval is removed in the pth operation but the intervals left after the $(p-1)$th operation are subdivided in two equal parts, and we deal in the $(p+1)$th operation, with the intervals thus subdivided.

[3] See Menchoff [1]; Zygmund [1, p. 294].

and $F_p(x)$ linear in each η_{pk}. The consideration, in an interval η_{pk}, of the straight line representing F_p and of the broken line representing F_{p+1} shows immediately that

$$|F_p - F_{p+1}| \leqq \frac{1}{2^{p+1}} \frac{\frac{1}{2}\eta_p - \eta_{p+1}}{\frac{1}{2}\eta_p};$$

that is to say, by (1)

$$|F_p - F_{p+1}| \leqq \frac{1}{2^{p+1}}\left[1 - \frac{E(p+1)}{E(p)}\right]. \tag{2}$$

Thus F_p converges uniformly to a continuous function F which is non-decreasing, is constant in every interval contiguous to P but increasing from one interval to another, and we have $F(0)=0$, $F(2\pi)=1$.

Let us consider now the Fourier-Stieltjes coefficients of dF. We have

$$c_n = \int_0^{2\pi} e^{nix} dF = \int_0^{2\pi} e^{nix} d(F - F_p) + \int_0^{2\pi} e^{nix} dF_p.$$

Denoting by A and B the two last integrals, and integrating A by parts, we have, as $F - F_p$ vanishes for $x=0$ and $x=2\pi$,

$$|A| \leqq n \int_0^{2\pi} |F - F_p| \, dx.$$

Hence

$$|A| \leqq n \int_0^{2\pi} |F_p - F_{p+1}| \, dx + n \int_0^{2\pi} |F_{p+1} - F_{p+2}| \, dx + \cdots.$$

But

$$\int_0^{2\pi} |F_p - F_{p+1}| \, dx \leqq E(p) \cdot \max |F_p - F_{p+1}|,$$

since $F_p = F_{p+1}$ when x does not belong to E_p. Hence, by (2)

$$\int_0^{2\pi} |F_p - F_{p+1}| \, dx \leqq \frac{1}{2^{p+1}} [E(p) - E(p+1)].$$

Let us suppose now that $E(p)$ is a convex function; then $E(p) - E(p+1)$ is decreasing and we have

$$|A| \leqq n[E(p) - E(p+1)] \sum_{\nu=p}^{\infty} \frac{1}{2^{\nu+1}} = \frac{n}{2^p} [E(p) - E(p+1)].$$

Considering now the integral B we have

$$\int_{\eta_{pk}} e^{niz}dF_p = \frac{1}{2^p\eta_p}\int_{\eta_{pk}} e^{niz}dx = \frac{1}{E(p)}\int_{\eta_{pk}} e^{niz}dx.$$

Hence

$$|B| \leqq \frac{2^p}{E(p)}\cdot\frac{2}{n}$$

and

$$|c_n| \leqq \frac{n}{2^p}[E(p) - E(p+1)] + \frac{2^{p+1}}{n}\frac{1}{E(p)}. \tag{3}$$

Up to here we have followed very closely the above quoted Menchoff's example, in extending it to general symmetrical sets. In Menchoff's example

$$\xi_p = \frac{p}{2(p+1)}, \qquad E(p) = \frac{1}{p+1}.$$

Now the inequality (3) gives us in a very simple way a class of symmetrical perfect sets of measure zero which are sets of multiplicity. As is well known, in order that P should be a set of multiplicity it is sufficient to have $c_n = o(1)$.

The integer p being arbitrary, let us choose it satisfying the following condition:

$$\frac{2^{p-1}}{(E(p-1))^{1/2}(E(p-1) - E(p))^{1/2}} < n \leqq \frac{2^p}{(E(p))^{1/2}(E(p) - E(p+1))^{1/2}}. \tag{4}$$

Thus, by (3)

$$|c_n| \leqq \left(1 - \frac{E(p+1)}{E(p)}\right)^{1/2} + 4\frac{E(p-1)}{E(p)}\left(1 - \frac{E(p)}{E(p-1)}\right)^{1/2}$$

If n increases infinitely, so does p defined by (4); hence if we suppose $E(p+1)/E(p) \to 1$, we have $c_n = o(1)$.

We have thus obtained the following result:[4] *Each symmetrical perfect set of measure zero such that $E(p)$ is convex and $E(p+1)/E(p)$ tends to 1 is a set of multiplicity.*

[4] A more general result is due to Nina Bary [1], but the proof is very complicated

2. **The Fourier-Stieltjes coefficients of singular monotone functions.** Let us now consider for every p the characteristic function f_p of the set E_p. We have

$$\int_0^{2\pi} e^{niz} dF_p = \frac{1}{2^p \eta_p} \int_{E_p} e^{niz} dx = \frac{1}{E(p)} \int_0^{2\pi} f_p(x) e^{niz} dx.$$

Hence

$$c_n = \int_0^{2\pi} e^{niz} dF = \frac{1}{E(p)} \int_0^{2\pi} f_p(x) e^{niz} dx + A,$$

A denoting, as above, the integral

$$\int_0^{2\pi} e^{niz} d(F - F_p).$$

We have seen that

$$|A| \leqq n \sum_{\nu=p}^{\infty} \frac{1}{2^{\nu+1}} [E(\nu) - E(\nu+1)].$$

Here we shall make no hypothesis whatever about the behaviour of $E(p) - E(p+1)$. It is plain that in any case we have

$$|A| \leqq \frac{nE(p)}{2^{p+1}}.$$

Hence

$$E^2(p) |c_n|^2 \leqq 2 \left| \int_0^{2\pi} f_p(x) e^{niz} dx \right|^2 + 2 \left(\frac{nE^2(p)}{2^{p+1}} \right)^2. \tag{5}$$

The integer p is arbitrary. We can associate to every n an integer $p = p(n)$ non-decreasing, tending to infinity with n, and sum the inequalities thus obtained from $n = 1$ to $n = \infty$ provided that the sum of the right-hand side terms be convergent.

Let us observe that the sets E_p are such that $E_{p+1} \subset E_p$ that is to say $f_{p+1} \leqq f_p$. Now it is known that in this case the sum

$$S = \sum_1^{\infty} \left| \int_0^{2\pi} f_p(x) e^{niz} dx \right|^2,$$

where p takes the same value for all n such that $n_p \leqq n < n_{p+1}$, converges if $n_{p+1}/n_p > \lambda > 1$; moreover $S < C(\lambda)$, C being a constant depending only on λ. This is a consequence of the well known theorem stating that if $S_n(x)$ is the nth sum of the Fourier series of a function ϕ of integrable square, then

$$\int_0^{2\pi} \sup_p |S_{n_p}(x)|\,dx \leqq K(\lambda)\left[\int_0^{2\pi} \phi^2 dx\right]^{1/2}$$

where $K(\lambda)$ depends on λ only.[5]

Hence if we take, for example $p=[a \log n]$ ($[m]$ denoting the integral part of m), p will take the same value for $n_p \leqq n < n_{p+1}$, n_p being given by

$$a \log (n_p - 1) < p \leqq a \log n_p.$$

Hence

$$a \log n_{p+1} - a \log (n_p - 1) > 1;$$

that is to say,

$$\frac{n_{p+1}}{n_p} > \lambda > 1$$

λ depending on a only.

On the other hand, we will have

$$\frac{n}{2^{p+1}} < \frac{n}{2^{a \log n}}$$

and it is easily seen that the series $\sum(n/2^{p+1})^2$ will converge if $a>3/(2 \log 2)$.

Hence, returning to inequality (5) and writing $E[a \log n]$ instead of $E([a \log n])$, we get the following result:

The series

$$T = \sum_1^\infty E^2[a \log n] \cdot |c_n|^2$$

is convergent for $a>3/(2 \log 2)$. *Moreover* $T<C(a)$, *C being a constant depending on a only.*

[5] See Zygmund [1, p. 252]; and for the equivalence of the two statements, Salem [1, 2]. The proof is briefly as follows: $p=p(x)$ being any function of x, we have

$$\left|\int_0^{2\pi} S_{n_{p(x)}}(x)dx\right| \leqq K(\lambda)\left[\int_0^{2\pi} \phi^2 dx\right]^{1/2},$$

and this can be written if $\phi(x) \sim \sum c_n e^{nix}$,

$$\left|\sum c_n \int_0^{2\pi} f_p e^{nix} dx\right| \leqq K_1(\lambda)(\sum |c_n|^2)^{1/2},$$

where the f_p are characteristic functions of sets, corresponding to $p(x)$, such that $f_{p+1} \leqq f_p$ and such that f_p is the same for all n for which $|n_p| \leqq |n| < |n_{p+1}|$. Since this holds good for any sequence $\{c_n\}$ such that $\sum |c_n|^2 < \infty$, the statement about the sum S follows immediately.

The interest of this theorem lies in the fact that we can construct our perfect set P of measure zero such that $E(p)$ tends to zero as slowly as we please. Hence, while the function $F(x)$'s being singular (that is, constant in every interval contiguous to P) makes the series $\sum|c_n|^2$ necessarily divergent, the divergence of this series can be as slow as we please, and the rapidity of the divergence is connected in a very simple way with the function $E(p)$.

We refer the reader to other results which have been obtained previously in this field. See Wiener and Wintner [1, 2] and Schaeffer [1].

3. **Absolute convergence of trigonometrical series.** The set P is said to be of the type N if there exists a trigonometrical series $\sum\rho_n\cos(nx-\alpha_n)$ with $\rho_n\geqq 0, \sum\rho_n=\infty$, such that $\sum\rho_n|\cos(nx-\alpha_n)|<\infty$ for every x belonging to P. It has been proved[6] that if P is of the type N we have

$$\limsup\int_0^{2\pi}\cos 2nx\,dF=1.$$

If we split the sequence of integers $\{n\}$ in two complementary sequences $\{m_k\}$, $\{n_k\}$ such that for every k:

$$(6)\qquad \int_0^{2\pi}\cos 2m_kx\,dF\leqq 1-\epsilon,\qquad \int_0^{2\pi}\cos 2n_kx\,dF>1-\epsilon,$$

ϵ being any fixed positive number <1, the proof of the theorem just stated shows that no series $\sum r_k\cos(m_kx-\alpha_k)$ can converge absolutely in P unless $\sum r_k<\infty$. Hence, if the set P is of the type N every trigonometrical series absolutely convergent in P but not everywhere can be broken into two complementary series: the first one,

$$\sum r_k\cos(m_kx-\alpha_k)\qquad\text{with }\sum r_k<\infty,$$

which is a parasitic series convergent absolutely everywhere; and the second one, $\sum\rho_k\cos(n_kx-\beta_k)$, which converges absolutely in P, with $\sum\rho_k=\infty$.

Now, the question arises of what can be the frequency of the integers $\{n_k\}$. We can draw some conclusions from the theorem proved in §2. We have, by (6), by a suitable choice of ϵ, $|c_{2n_k}|>\frac{1}{2}$. Hence

$$\sum_1^\infty E^2[a\log 2n_k]<\infty.$$

[6] See Salem [3, p. 323].

Roughly speaking, the more slowly $E(p)$ decreases, the more sparse are the integers $\{n_k\}$. More precisely if between p and q there are $\nu(p, q)$ integers of the sequence $\{n_k\}$ we have

$$\nu \cdot E^2[a \log 2q] < \sum_{p<n_k<q} E^2[a \log 2n_k] < E[a \log 2p] \cdot C(a);$$

hence

$$\nu < C(a) \cdot \frac{E[a \log 2p]}{E^2[a \log 2q]}. \tag{7}$$

It has been shown[7] that the perfect set P for which all the ξ_p are equal to $\frac{1}{2}$, except for a sequence $\{i_j\}$ for which $\xi_{i_j} = 1/2j$, is of the type N. By taking the sequence $\{i_j\}$ very sparse, we can plainly obtain a set of the type N with $E(p)$ decreasing as slowly as we please. Hence, the inequality (7) shows that there exist sets of the type N for which $\nu[p, \phi(p)] < \omega(p)$, $\phi(p)$ increasing as rapidly, and $\omega(p)$ as slowly as we please.

This result throws some light on the fact that the sum of two sets P, P', both of the type N, may be not of the type N, a result which is due to Marcinkiewicz.[8] If the sets P, P' are both of the type N, then, F' being the corresponding function for the set P', there exist two sequences of integers $\{n_k\}$, $\{n_k'\}$ such that

$$\lim \int_0^{2\pi} \sin^2 n_k x \, dF = 0, \qquad \lim \int_0^{2\pi} \sin^2 n_k' x \, dF' = 0 \tag{8}$$

by the theorem stated at the beginning of §3. But it can be shown exactly in the same way that if $P+P'$ is of the type N, we must have

$$\liminf \left[\int_0^{2\pi} \sin^2 nx \, dF + \int_0^{2\pi} \sin^2 nx \, dF' \right] = 0;$$

that is to say, there must exist a *common* sequence $\{m_k\}$ such that

$$\lim \int_0^{2\pi} \sin^2 m_k x \, dF = 0, \qquad \lim \int_0^{2\pi} \sin^2 m_k x \, dF' = 0. \tag{9}$$

Now the sequences $\{n_k\}$, $\{n_k'\}$ satisfying (8) can be, as we have seen, both very sparse, hence the existence of a common sequence $\{n_k\}$ satisfying (9) may be impossible.

The question of what happens if the second set P' is reduced to a

[7] See Salem [3, p. 329].

[8] Marcinkiewicz [1].

single point has been mentioned to me by Dr. Marcinkiewicz and Professor Zygmund. We will show here that *the sum of any set of the type N* (not necessarily a perfect set) *and of a single point is also of the type N.*

It has been proved[9] that by a suitable translation of the given set E we can suppose that the series converging absolutely in E is a sine series. Let us suppose, then, that for every x belonging to E we have

$$\sum_1^\infty \rho_n |\sin nx| < \infty, \qquad \rho_n \geqq 0, \ \sum \rho_n = \infty.$$

We can always find a positive function $\omega(n)$ increasing infinitely with n, such that

$$\sum_1^\infty \frac{\rho_n}{\omega(n)} = \infty, \qquad \sum_1^\infty \frac{\rho_n}{\omega^2(n)} < \infty.$$

(We can take, for example, $\omega(n) = \sum_1^n \rho_n$.) Let Q be a point of abscissa x_0, not belonging to E. By Dirichlet's theorem, for every n we can find an integer p_n such that

$$(10) \qquad 1 \leqq p_n \leqq \omega(n), \qquad \left| p_n\left(n\frac{x_0}{\pi}\right) - h_n \right| < \frac{1}{[\omega(n)]},$$

p_n being an integer and $(z) = z - [z]$ denoting the fractional part of z. Thus

$$\left| p_n n \frac{x_0}{\pi} - p_n\left[n\frac{x_0}{\pi}\right] - h_n \right| < \frac{1}{[\omega(n)]};$$

that is to say,

$$|p_n n x_0 - k_n \pi| < \frac{4}{\omega(n)},$$

k_n being an integer. Thus

$$|\sin p_n n x_0| < \frac{4}{\omega(n)}$$

for every n. Hence

$$(11) \qquad \sum_1^\infty \frac{\rho_n}{\omega(n)} |\sin p_n n x_0| < 4 \sum_1^\infty \frac{\rho_n}{\omega^2(n)} < \infty.$$

[9] See Salem [3, p. 319].

On the other hand, applying the well known theorem: $|\sin mt| \leqq m|\sin t|$ for m a positive integer, we have, by (10), if x belongs to E:

$$\sum_1^\infty \frac{\rho_n}{\omega(n)} |\sin p_n nx| \leqq \sum_1^\infty \frac{p_n}{\omega(n)} \rho_n |\sin nx| \leqq \sum_1^\infty \rho_n |\sin nx| < \infty. \tag{12}$$

Hence, by (11) and (12) the series $\sum_1^\infty [\rho_n/\omega(n)] \sin nx$ converges absolutely in $E+Q$. But $\sum_1^\infty \rho_n/\omega(n) = \infty$; hence $E+Q$ is of the type N.

This leads immediately to the following result: *The sum of any set of the type N and of any finite set is also of the type N.*

Bibliography

N. Bary

1. *Sur l'unicité du développement trigonométrique*, Fundamenta Mathematicae, vol. 9 (1927), pp. 62–118.

J. Marcinkiewicz

1. *Quelques théorèmes sur les séries et les fonctions*, Travaux de la Société des Sciences et des Lettres de Wilno, Classe des Sciences Mathématiques et Naturelles, vol. 12 (1937).

D. Menchoff

1. *Sur l'unicité du développement trigonométrique*, Comptes Rendus de l'Académie des Sciences, Paris, vol. 163 (1916), pp. 433–436.

R. Salem

1. *Sur une méthode de sommation, valable presque partout, pour les séries de Fourier de fonctions continues*, Comptes Rendus de l'Académie des Sciences, Paris, vol. 205 (1937), p. 14.
2. *Essais sur les Séries Trigonométriques*, Paris, Hermann, 1940.
3. *The absolute convergence of trigonometrical series*, Duke Mathematical Journal, vol. 8 (1941), pp. 317–334.

A. C. Schaeffer

1. *The Fourier-Stieltjes coefficients of a function of bounded variation*, American Journal of Mathematics, vol. 61 (1939), pp. 934–940.

N. Wiener and A. Wintner

1. *Fourier-Stieltjes transforms and singular convolutions*, American Journal of Mathematics, vol. 60 (1938), pp. 513–532 and Bibliography.

A. Zygmund

1. *Trigonometrical Series*, Warsaw, 1935.

Montreal, Quebec, Canada